MS-A0402 / Period IV 2023

Final Exam, 21.04.2023 - SOLUTIONS

No calculators or notes of any kind are allowed.

This exam consists of 5 problems, each of equal weight.
All answers must be justified with appropriate reasoning and calculations.

Notation:

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$
$$\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \,|\, x \geq 0\}$$
a divides b is denoted $a|b$

Question 1. Let P and Q be propositions

(a) Give one example of a sentence which is a proposition. Give one example of a sentence which is not a proposition. These can be mathematical or any other type of sentences.

Solution: Some examples of propositions: Two plus two equals four. Two plus two equals five. Right now the sky is blue.$\forall x, 3 + x = 7$. Some examples which are not propositions: Does two plus two equal 4? I always tell lies. $3 + x = 7$ (does not have truth value unless x is specified or there is a quantifier).

(b) If P is false and Q is true, what is the truth value of $P \implies Q$. This often confuses people when they first learn about "$\implies$". Explain why this is the right way to define "$\implies$" with the help of an example (in mathematical or everyday language) of propositions P and Q.

Solution 1: P="2 + 2 = 5", Q = "1 + 1 = 2". We can obtain Q from P by multiplying both sides of P by zero and then adding 1 to both sides twice.

Solution 2: P="It is Monday", Q ="I will wear a hat". Then $P \implies Q$ is the same as the statement "If it is Monday then I will wear a hat". Now if today is Tuesday and I am wearing a hat am I a truth teller or a liar. In my if ..then.. statement I did not claim anything about Tuesday. So no matter if I wear a hat or not on Tuesday I was telling the truth.

(c) Determine if $(P \implies Q) \vee (Q \implies P)$ is a tautology or not by using a truth table. Be sure to state your conclusion.

Solution:

P	Q	$(P \implies Q)$	$(Q \implies P)$	$(P \implies Q) \vee (Q \implies P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The proposition $(P \implies Q) \vee (Q \implies P)$ is thus a tautology as it is true in all cases, regardless of the truth value of the individual propositions P and Q.

Question 2. For each statement below, indicate if it is TRUE(T) or FALSE(F). No justification is needed. Write your answers in your exam booklet in a table like this

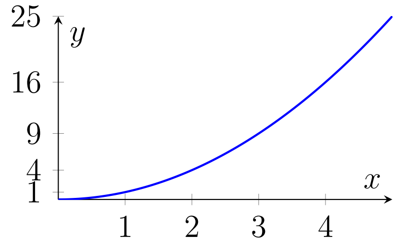
	a	b	c	d	e	f	g	h	i	j
T/F	F	T	F	F	T	T	F	F	F	T

(a) $f : \{1, 2, 3\} \to \{1, 3\}$ defined by $f(x) = x$ is a function.

F: f is not defined at all elements in the domain. In particular $f(2)$ is not defined

(b) $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $f(x) = x^2$ is a bijection.

T: The function is a bijection because $y = x^2$ has a unique solution for all $y \in \mathbb{R}_{\geq 0}$.



Also we can see that x^2 has a well-defined inverse $\sqrt{x}$ for $x \geq 0$.

(c) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$ is a bijection.

F: Not surjective. For example there is no element that maps to 3. That is $f(x) = x^2 = 3$ does not have a natural number solution.

(d) The cardinalities of $\mathbb{N}$ and $\mathbb{Z}$ are different because $\mathbb{Z}$ is essentially double $\mathbb{N}$.

F: As discussed in class these have the same cardinality. For example here is a bijection

$$f(n) = \begin{cases} -\frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

(e) The cardinalities of $\mathbb{Q}$ and $\mathbb{R}$ are different even though every real number can be approximated to arbitrary accuracy by rational numbers.

T: We discussed Cantor's diagonal argument in class that proves $\mathbb{Q}$ and $\mathbb{R}$ have different cardinalities. The ideas of the proof was to show that there can be no surjection.

(f) Let A be a finite set with cardinality $|A| = n$.

$|\mathcal{P}(A \times A)| =$ " the cardinality of the power set of the cartesian product $A \times A$" $= (2^n)^n$?

T: $|\mathcal{P}(A \times A)| = 2^{|A \times A|} = 2^{(n^2)} = 2^{(n \cdot n)} = (2^n)^n$

(g) Let $P(x)$ and $Q(y)$ be propositions depending on variables x and y respectively.

The negation of $\forall x \exists y (P(x) \implies Q(y))$ is $\exists y \forall x (P(x) \wedge \neg Q)$.

F: $\forall x R(x)$ negates to $\exists x \neg R(x)$.

(h) Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if $x^2 \leq y^2$.

The relation $\sim$ defines a partial order on $\mathbb{R}$.

F: If $x = -1$ and $y = 1$ then $x^2 \leq y^2$ and $y^2 \leq x^2$, but $x \neq y$. So the relation is not antisymmetric.

(i) If A and B are sets such that $A \times B = B \times A$ then $A = B$

F: This was from the homework. If A is any set and $B = \emptyset$ then $A \times B = B \times A = \emptyset$.

(j) Let A and B be finite sets. If $|B| > |A|$ then there exists an injection $f : A \to B$.

T: See the homework. This is difficult to prove in general, but for finite sets it is easier.

Question 3. (a) State the inclusion-exclusion principle for 3 sets.

Solution: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

(b) How many 7 digit numbers in $\mathbb{N}$ start or end with the digits 21. For example $\mathbf{21}56973$, $29827\mathbf{21}$.

Solution: There are 10^5 numbers that start with 21 and 9×10^4 numbers that end in 21 because numbers can't start with a zero (only one point was taken off in case this was missed). There are 10^3 numbers both starting and ending with 21. Using inclusion-exclusion for 2 sets, the answer is $100000 + 90000 - 1000 = 189000$

(c) Use induction to prove that for all $n \in \mathbb{N}$, $n \geq 1$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution: To prove the formula using induction, we'll start by verifying the base case when n = 1:

When n = 1:

$$\sum_{k=1}^{1} k^2 = 1^2 = 1$$

And

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1$$

The formula holds for $n = 1$.

Next, we'll assume that the formula holds for some arbitrary positive integer k, and then prove that it also holds for $k + 1$.

Assuming the formula holds for k:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Now, let's consider the sum for $k+1$:

$$\sum_{i=1}^{k+1} i^2 = (k+1)^2 + \sum_{i=1}^{k} i^2$$

Now use the induction assumption.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \\
&= \frac{6(k+1)^2 + k(k+1)(2k+1)}{6} \\
&= \frac{(k+1)(6(k+1) + k(2k+1))}{6} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

We have the expression in the same form as the original formula, but with $k+1$ substituted for n. Thus, by induction, we have shown that the formula holds for all $n \geq 1$. Note: It's probably easier to expand the above expression and compare with the expansion of what we are trying to show with $n = k+1$, but factoring is more fun!

Therefore, the equation

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

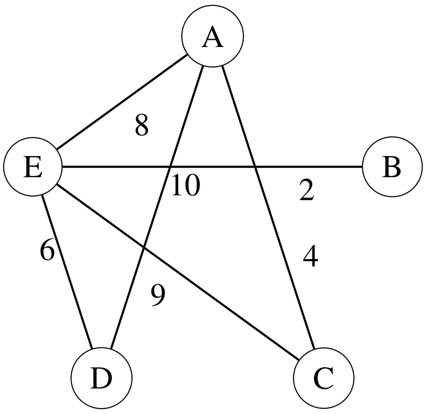
holds for all $n \geq 1$.

(d) Consider the permutation $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 1 & 8 & 3 & 4 & 7 \end{pmatrix}$. Write ρ^2 as a product of disjoint cycles.

solution: $\rho^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 3 & 6 & 5 & 7 & 2 & 1 & 4 \end{pmatrix} = (18457)(236)$

Question 4. Consider the weighted graph $G = (V, \mathcal{E}, w)$ where the vertex set is $V = \{A, B, C, D, E\}$, the edge set is $\mathcal{E} = \{\{A, C\}, \{A, D\}, \{A, E\}, \{B, E\}, \{C, E\}, \{D, E\}\}$ and the weights are $w(\{A, C\}) = 4, w(\{A, D\}) = 10, w(\{A, E\}) = 8, w(\{B, E\}) = 2, w(\{C, E\}) = 9, w(\{D, E\}) = 6$.

(a) Make a large clear sketch of the graph. It is suggested that you draw it as a regular pentagon.



(b) Find a minimal spanning tree of G using a greedy algorithm. Your answer can be a sketch. Briefly explain each step in your selection of edges.

Solution:

To find a minimum spanning tree (MST) using Prim's algorithm, we: (1) Initialize a tree with a single vertex, chosen arbitrarily from the graph. (2) Grow the tree by one edge: Of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and transfer it to the tree. (3) Repeat step 2 until all vertices are in the tree.

Let's go step by step:

Step 1: Start with vertex A as the initial tree.

Step 2: Look for the minimum weight edge that connects the current tree to a vertex outside the tree. Add that vertex and the edge to the MST. Minimum weight edge is A,C. MST: {AC} (just writing the edges and not the vertices)

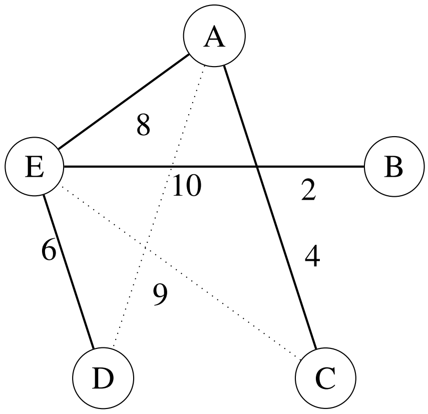
Step 3: Minimum weight edge is A,E. MST: {AC, AE}

Step 4: . Minimum weight edge is B,E. MST: {AC, AE, BE}

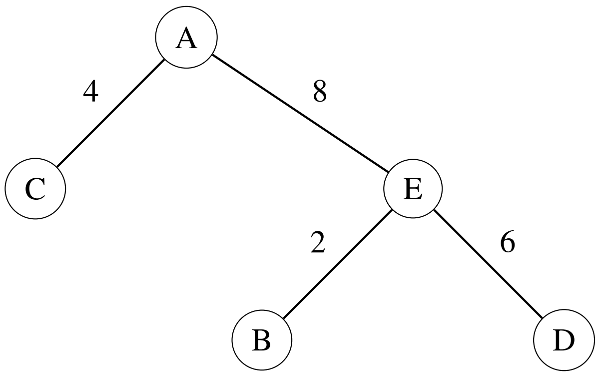
Step 5: Minimum weight edge is D,E. MST: {AC, AE, BE, DE}

Step 6: All vertices have been included in the MST, so the algorithm is complete.

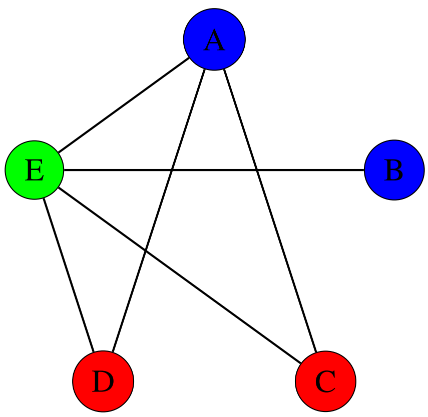
The final MST starting at vertex A is: MST: {AC, AE, BE, DE}



Or, redrawing to look more like a tree.



(c) Use a greedy algorithm to find a (vertex) colouring of G. Your answer can be a sketch.



(d) What is the chromatic number of G. Justify your answer.

We can see that vertices A, C, and E form a complete subgraph K3 (a triangle), as they are pairwise adjacent. Thus, we have an instance of K3 within graph G.

Therefore, since graph G contains a subgraph isomorphic to K3, we can conclude that the chromatic number of graph G is at least 3 (lower bound). Additionally, since we were able to color graph G using three colors, we have shown that the chromatic number is at most 3 (upper bound).

Hence, we can state that the chromatic number of graph G is exactly 3.

Question 5. (a) Find all solutions to the linear Diophantine equation: $60x + 33y = 9$.

Method A: Here is the standard but long solution.

To find all solutions to the Diophantine equation $60x + 33y = 9$, we will use the extended Euclidean algorithm. The first step is to find the greatest common divisor (GCD) of 60 and 33.

Step 1: Find the GCD of 60 and 33.

Using the Euclidean algorithm:

$$\begin{aligned}
60 &= 1 \times 33 + 27 \\
33 &= 1 \times 27 + 6 \\
27 &= 4 \times 6 + 3 \\
6 &= 2 \times 3 + 0
\end{aligned}$$

So the GCD of 60 and 33 is 3. Since $3|9$ we know there is a solution to the Diophantine equation.

Step 2: Express the GCD (3) as a linear combination of 60 and 33.

Using the Euclidean algorithm backward:

$$\begin{aligned}
3 &= 27 - 4 \times 6 \\
&= 27 - 4 \times (33 - 27) \\
&= 5 \times 27 - 4 \times 33 \\
&= 5 \times (60 - 33) - 4 \times 33 \\
&= 5 \times 60 - 9 \times 33
\end{aligned}$$

Therefore $15 \times 60 - 27 \times 33 = 9$.

Thus, we have found one solution to the equation $60x + 33y = 3$, which is $x_0 = 15$ and $y_0 = -27$.

Step 3: Find all solutions by adding all solution of $60x + 33y = 0$ which, by reducing to the equivalent equation $20x + 11y = 0$ by dividing by the GCD, is seen to be $x = 11n$, $y = -20n$.

All solution of the Diophantine equation are

$$x = 15 + 11n$$
$$y = -27 - 20n$$

for $n \in \mathbb{Z}$.

Method B:

Dividing the equation by 3 we get $20x + 11y = 3$ which cannot be reduced further since 11 is a prime. To get the units digit to be 3, we see y has to end in a 3 or 7. We see that $x_0 = 4,_0 = -7$ is a solution. The solutions to $20x + 11y = 0$, is $x = 11n$, $y = -20n$. So the general solution to the Diophantine equation is $x = 15 + 11n, y = -27 - 20n$ for $n \in \mathbb{Z}$.

(b) Depending how you solve this problem it may be useful to recall that

- $a|b$ is an order relation.
- Let p be a prime and $a_i \in \mathbb{Z}$. If $p|a_1a_2 \cdots a_n$ then $p|a_1 \vee p|a_2 \vee \cdots \vee p|a_n$. A special case is that if $p|ab$ then p divides a or p divides b.

Let p be a prime and $n \in \mathbb{N}$. Prove that the set of divisors of p^n is $\{1, p, p^2, \ldots p^{n-1}, p^n\}$.

Solution:

Let D be the let of divisors of p^n. We need to show (A) $D \subseteq \{1, p, p^2, \ldots p^{n-1}, p^n\}$ and, (B) $D \supseteq \{1, p, p^2, \ldots p^{n-1}, p^n\}$.

(A) For each $i = 1, \ldots n$ we can write $p^n = p^i p^{n-i}$. Since p^i divides the right-hand side, it also divides the left. So $p^i \in D$ for $i = 1, \ldots, n$, and thus we have shown (A).

(B) Let $q \in D$. By the fundamental theorem of arithmetic, q can be written as a product of primes. That is $q = p_1 \cdot p_2 \cdots p_m$. Now for each $k = 1 \ldots m$, $p_k|q$. Since $q|p^n$, by transitivity of divisibility order relation, we have $p_k|p^n$ and by applying the given theorem we have $p_k|p$. So $p_k = 1$ or p. Thus $q = p^r$, and $r \leq n$ as $q \leq p^n$. So we have shown (B) which completes the proof.

(B) can also be proved by invoking the uniqueness of prime factorization: As the factorization of p^n is $p^n = p \cdot p \cdots p$, there can be no other prime factor of p^n except p. Then use the argument in (B) above for any q that divides p^n.

Many people tried as inductive proof, which is possible but it requires the same argument as in step (B), so is of no advantage.