Lecture 4

Recall:

Let C be a general projective cubic defined over K by

$F(X,Y,Z) = aX^3+bX^2Y+cXY^2+dY^3+eZX^2+fZXY+gZY^2+hZ^2X+iZ^2Y+jZ^3=0.$

Suppose $O \in C(K)$ is not an inflection point.

$\Rightarrow T_O C$ meets C in another point $Q \in C(K)$.

Consider $T_Q C$, which meets in another $R \in C(K)$

Take $L \neq T_O C$, a line by O, which meets $T_Q C$ at $P \in C(K)$ and meets C at $S \in C(\overline{K})$.

Consider the reference system $\{O=(1:0:0), Q=(0:1:0), P=(0:0:1)\}$.

Since $O, Q \in C \Rightarrow a=d=0$. We get:

$C: bX^2Y+cXY^2+eZX^2+fZXY+gZY^2+hZ^2X+iZ^2Y+jZ^3=F(X,Y,Z)=0.$

In the new reference, $T_O C$ is $Z=0$

$T_Q C$ is $X=0$

L is $Y=0$.

$\Rightarrow \left(\frac{\partial F}{\partial X},\frac{\partial F}{\partial Y},\frac{\partial F}{\partial Z}\right)(1:0:0)=(0,b,e) \Rightarrow T_O C: bY+eZ=0 \Rightarrow b=0 \Rightarrow$

$C: cX^2Y+eX^2Z+fZXY+gZY^2+hZ^2X+iZ^2Y+jZ^3=0.$

Since $T_Q C$ is $X=0 \Rightarrow$ it cuts C in only one new point $\underset{(0:a:b)}{S}$

$\left.\begin{array}{l} X=0 \\ F(X:Y:Z)=0 \end{array}\right\} \Rightarrow gZY^2+iZ^2+jZ^2=0$ has only 1 solution (affine)

$\Rightarrow g=0.$

In the affine plane $(x = \frac{X}{Z}, y = \frac{Y}{Z})$ we are left with

$$C: \; cxy^2 + ex^2 + fxy + hx + iy + j = 0 \qquad \text{or:}$$

$$xy^2 + (ax+b)y = cx^2 + dx + e$$

$$C \longrightarrow C'$$
$$(x,y) \longmapsto (x, yx)$$

C' is birationally equivalent to C and has equation

$$(xy)^2 + (ax+b)xy = cx^3 + dx^2 + ex \qquad \text{or:}$$

$$y^2 + (ax+b)y = cx^3 + dx^2 + ex$$

Setting $(\bar{x}, \bar{y}) = (x, y - \frac{1}{2}(ax+b))$ we get

$$y^2 = \lambda x^3 + \mu x^2 + \nu x + \delta$$

Setting $(x,y) = (\lambda\bar{x}, \lambda^2\bar{y})$, we get

$$\boxed{C: \; y^2 = x^3 + Ax + B}$$

Def: $\Delta_C = -16(4A^3 + 27B^2)$

$$j_C = -1728(4A)^3 / \Delta_C.$$

- Isomorphisms of curves

Def: Let $\bar{C}_1, \bar{C}_2$ be projective plane curves; C_1 and C_2 their affine avatars. $\bar{C}_1$ is isomorphic to $\bar{C}_2$ if $\exists\, \phi: \mathbb{P}^2(K) \to \mathbb{P}^2(K)$ given by

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{M} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \quad \text{s.t. } \phi(\bar{C}_1) = \phi(\bar{C}_2)$$

$, M \in PGL_2(K) = GL_3(K)/N$

ϕ induces an affine isomorphism $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$

where $\begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} \mapsto \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \Rightarrow \phi(C_1) = C_2$.

Prop: the only changes of variable which preserve Weierstrass forms are

$$\begin{cases} x = u^2 \bar{x} \\ y = u^3 \bar{y} \end{cases}, u \in \bar{K}.$$

Proof: Let $(x = a\bar{x} + b\bar{y} + c,\ y = d\bar{x} + e\bar{y} + f)$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ respecting

Weierstrass $\Rightarrow$

$$y^2 = d^2\bar{x}^2 + e^2\bar{y}^2 + 2de\,\bar{x}\bar{y} + 2df\bar{x} + 2ef\bar{y} + f^2 = (a\bar{x} + b\bar{y} + c)^3 + A(a\bar{x} + b\bar{y} + c) + B$$

$$= a^3\bar{x}^3 + b^3\bar{y}^3 + 3ab^2 x\bar{y}^2 + 3a^2 b\bar{x}^2\bar{y} + 6abc\,\bar{x}\bar{y} + 3b^2 c\bar{y}^2 + 3a^2 c\bar{x}^2 + b^3 y^3 +$$

$$+ \underbrace{(3c^2 + A)}_{A'} a\bar{x} + \underbrace{(3c^2 + A)}_{\overset{\shortparallel}{0}} b\bar{y} + \underbrace{(Ac + B + c^3)}_{B'} \Rightarrow \boxed{b = 0} \Rightarrow d \neq 0$$

$$d^2\bar{x}^2 + e^2\bar{y}^2 + 2de\,\bar{x}\bar{y} + 2df\bar{x} + 2ef\bar{y} + f^2 = a^3 x^3 + 3a^2 c\bar{x}^2 + (3c^2 + A)a\bar{x} + (Ac + B + c^3)$$

$\Rightarrow e \neq 0$, $de = 0 \Rightarrow \boxed{d = 0}$

$$e^2\bar{y}^2 + 2ef\,\bar{y} + f^2 = a^3 x^3 + 3a^2 c\,\bar{x}^2 + A'x + B'$$

$e \neq 0$, $ef = 0 \Rightarrow f = 0$

$$e^2\bar{y}^2 = a^3 x^3 + 3a^2 c\,\bar{x}^2 + A'x + B' \Rightarrow \boxed{c = 0}$$

hence, the change is $(x = a\bar{x},\ y = e\bar{y})$

$$y^2 = e^2\bar{y}^2 = (e\bar{y})^2 = a^3\bar{x}^3 + A'\bar{x} + B' = (a\bar{x})^3 + \frac{A'}{a}(\bar{x}a) + B'$$

$$\Rightarrow \bar{y}^2 = \frac{a^3}{e^2}\bar{x}^3 + \frac{A'}{e^2}\bar{x} + B' \quad (A)$$

Now $(\bar{x},\bar{y}) = (\bar{x}, e^{-4} y_1)$ takes ⊛ to

$$e^{-8} y_1^2 = \frac{a^3}{e^2}\bar{x}^3 + \frac{A'}{e^2}\bar{x} + B' \equiv y_1^2 = a^3 e^6 \bar{x}^3 + \frac{A'}{a} e^6 a\bar{x} + B' e^8$$

$$\parallel$$

$$(a.e^2\bar{x})^3 + \frac{A'}{a} e^4 (a e^2 \bar{x}) + B' e^8$$

$$\parallel\!|$$

$$y_1^2 = x_1^3 + \frac{A'}{a} e^4 x_1 + B' e^8$$

$$(x,y) = (a\bar{x}, e\bar{y}) =$$

$$= (a\bar{x}, e \cdot e^{-4} y_1) = (a\bar{x}, e^{-3} \cdot y_1) = (a \frac{1}{ae^2} x_1, e^{-3} y_1) = (e^{-2} x_1, e^{-3} y_1) \#$$

Prop: Let $C: y^2 = x^3 + Ax + B$ an affine cubic. Then $\tilde{C}$ singular $\iff \Delta_C = 0$.

Proof: Notice, the proj model: $\underbrace{Y^2Z - X^3 - AXZ^2 - BZ^3}_{F(X,Y,Z)} = 0$
$\nabla F(0:1:0) = (0,0,1) \neq (0,0,0) \Rightarrow \mathcal{O}$ is smooth.

In the affine chart:
$\exists (x,y) \in K^2 \mid y^2 = x^3 + Ax + B$ singular $\iff$ $y = 0$, $f'(x) = 0$, $f(x) = 0$

$\Rightarrow 3x^2 + A = 0 \Rightarrow x = \pm i\sqrt{\frac{A}{3}} \in \bar{K}$.

Hence, C singular $\iff (\pm i\sqrt{\frac{A}{3}}, 0) \in C(\bar{K})$. This happens $\iff$

$-i\frac{A^{3/2}}{3^{3/2}} + i\frac{A^{3/2}}{3^{1/2}} + B = 0 \iff -i A^{3/2} + i\, 3 . A^{3/2} + B . 3^{3/2} = 2i A^{3/2} + B . 3^{3/2} = 0$

$\iff -4A^3 = 27B^2 \iff 4A^3 + 27B^2 = 0 \#$

Obs. $E: y^2 = x^3 + Ax + B$ / $\mathbb{Q}$ elliptic curve $A = \frac{p}{q}$, $B = \frac{r}{s}$ $\Rightarrow$

$y^2 = x^3 + \frac{p}{q}x + \frac{r}{s} \Rightarrow qsy^2 = qsx^3 + psx + qr \Rightarrow$

$\cdot qs$ $\qquad\qquad \cdot (qs)^5$

$(qs)^6 y^2 = (qs)^6 x^3 + p q^5 s^6 x + q^6 s^5 r$

$\parallel$

$(q^3 s^3 y)^2 = (q^2 s^2 x)^3 + p q^3 s^4 q^2 s^2 x + q^6 s^5 r \Rightarrow \bar{y}^2 = \bar{x}^3 + A'\bar{x} + B'$

$\parallel$

$p q^3 s^4 (q^2 s^2 x)$ $\qquad A', B' \in \mathbb{Z}$.

Def: E has "good reduction" at p if $p \nmid \Delta$ ($\Rightarrow E/\mathbb{F}_p$ elliptic curve).

Otherwise p has "bad reduction".

Prop: Let E_1, E_2 be 2 elliptic curves over an algebraically closed field (assume hence $O_{E_1} = O_{E_2}$). Then $E_1 \cong E_2 \Leftrightarrow j_{E_1} = j_{E_2}$.

Proof: $E_1: y^2 = x^3 + Ax + B$ $\qquad \Delta_{E_1} = -16(4A^3 + 27B^2) \neq 0$

$E_2: y^2 = x^3 + A'x + B'$ $\qquad \Delta_{E_2} = -16(4A'^3 + 27B'^2) \neq 0$.

$x = u^2 x'$, $y = u^3 y' \Rightarrow A' = Au^{-4}$, $B' = Bu^{-6}$

"$\Rightarrow$" If they are isomorphic

$\Rightarrow \Delta' = \Delta . u^{-12} \Rightarrow j_{E'} = j_E$.

"$\Leftarrow$" Supp $\dfrac{(4A)^3}{4A^3 + 27B^2} = \dfrac{(4A')^3}{4(A')^3 + 27(B')^2} \Rightarrow A^3 . (B')^2 = (A')^3 . B^2$

want $(x, y) = (u^2 x', u^3 y')$.

Case 1: $A = 0 \Rightarrow B \neq 0 \Rightarrow A' = 0 \Rightarrow j_E = j_{E'} = 0$.

Case 2: $B = 0 \Rightarrow A \neq 0 \Rightarrow B' = 0 \Rightarrow j_E = j_{E'} = -1728$.

Case 3: $AB \neq 0 \Rightarrow A'B' \neq 0$ otherwise, if $B' = 0 \Rightarrow A' = 0 \Rightarrow j_{E'} = 0$

hence $A^3 (B')^2 = (A')^3 B^2 \Rightarrow \left(\frac{A}{A'}\right)^3 = \left(\frac{B}{B'}\right)^2$. Take $u = \sqrt[4]{\frac{A}{A'}} = \sqrt[6]{\frac{B}{B'}}$

$$y^2 = x^3 + Ax + B \Rightarrow u^6 \bar{y}^2 = u^6 \bar{x}^3 + Au^2 \bar{x} + B \Leftrightarrow$$

$$\uparrow$$

$$(x, y) = (u^2 \bar{x}, u^3 \bar{y})$$

$$\bar{y}^2 = \bar{x}^3 + Au^{-4}\bar{x} + Bu^{-6} \qquad Au^{-4} = A', \quad Bu^{-6} = B'. \quad \#$$

Prop: For each $j_0 \in \bar{K}$, $\exists\, E/K(j_0)$ s.t. $j_E = j_0$.

Proof: $j_0 \neq 0, 1728$

$$E: y^2 + xy = x^3 - \frac{36}{j_0 - 1728} x - \frac{1}{j_0 - 1728}.$$

$j = 0$: $\quad y^2 + y = x^3 \qquad \Delta = -27$

$j = 1728$: $\quad y^2 = x^3 + x \qquad \Delta = -64$,

if $\mathrm{char}(K) = 2 \Rightarrow 0 = 1728$

if $\mathrm{char}(K) = 3 \Rightarrow 0 = 1728$.