Math Camp - Final Exam

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Daniel Hauser

The exam is 2 hours and has a total of 120 points. It is closed book, and calculators are prohibited. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Short questions:

   (a) *(5 points)* True or false, every set that is not open is closed. If true, provide a proof, if false, a counterexample.

   (b) *(5 points)* Show that any strictly concave function being maximized over a convex set has at most one maximizer.

   (c) *(5 points)* What is the expected value of a random variable with pdf $f(x) = \frac{4}{\pi}\frac{1}{1+x^2}$ on $[0, 1]$.

   (d) *(5 points)* Suppose X and Y are independent and both have pdfs e^{-x} on $[0, \infty)$. What is the joint distribution of $X + Y$ and Y.

2. Consider the following constrained maximization problem:

$$\max_{x,y \in \mathbb{R}^2_+} \sqrt{x} + \sqrt{y}$$
$$\text{s.t. } x^2 + y^2 \leq 9$$
$$\frac{x^2}{4} + \frac{y^2}{9} \geq 1$$

   (a) *(5 points)* For any non-zero x, y that satisfy the constraints, show that the Hessian matrix of the objective function is negative semi-definite.

   (b) *(15 points)* What are the KKT conditions for this problem? (feel free to disregard non-negativity constraints)

   (c) *(15 points)* Are there any points that satisfy the KKT conditions and have $x^2/4 + y^2/9 > 1$ and $x, y > 0$?

(d) *(15 points)* Consider the alternative problem

$$\max_{x,y\in\mathbb{R}^2_+} \sqrt{x}+\sqrt{y}$$
$$\text{s.t. } x^2+y^2\leq 9.$$

Are the KKT conditions necessary and sufficient here? What does this tell you about the point you found in c?

3. Consider the consumer problem

$$\max_{x,y\in\mathbb{R}^2_+} u(x,y)$$
$$\text{s.t. } p_1x+p_2y\leq m$$

$u(x,y)$ is twice continuously differentiable, strictly concave, and strictly increasing (i.e. if $(x,y)\geq(x',y')$ then $u(x,y)\geq u(x',y')$, strictly so if $(x,y)\neq(x',y')$).

(a) *(15 points)* What are the KKT conditions for this problem.

(b) *(5 points)* Argue that $p_1x+p_2y\leq m$ holds with equality at any maximum.

(c) *(10 points)* Let $v(p,m)$ be the value function for this problem (the utility achieved at the maximum) and $x(p,m)$ and $y(p,m)$ be the demands (the arg maxes) and $\lambda(p,m)$ be the corresponding multiplier. If non-negativity constraints do not bind at the maximum and $x(p,m)$ is differentiable, show that $\frac{\partial v}{\partial p_1}=-\lambda(p,m)x(p,m)$.

(d) *(20 points)* Assume $(x^*,y^*,\lambda^*)=(1,1,1)$ satisfies the Lagrange multiplier conditions, is the maximum at $(p_1,p_2,m)=(1,1,2)$ and

$$D^2u(1,1)=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

Solve for $\partial x/\partial m, \partial y/\partial m$ and $\partial\lambda/\partial m$ at $(p_1,p_2,m)=(1,1,2)$ (hint: what three equations must the maximum satisfy).