Lecture 1
Convex sets

- subspace, affine set, convex set, convex cone
- simple examples and properties
- combination and hulls
- ellipsoids, polyhedra, norm balls
- affine and projective transformations
- separating hyperplanes
- generalized inequalities

Subspaces

$S \subseteq \mathbf{R}^n$ is a *subspace* if

$$x, y \in S, \quad \lambda, \mu \in \mathbf{R} \implies \lambda x + \mu y \in S$$

Geometrically: $x, y \in S \Rightarrow$ plane through $0, x, y \subseteq S$

Representation

$$\begin{aligned} \mathrm{range}(A) &= \{Aw \mid w \in \mathbf{R}^q\} \\ &= \{w_1 a_1 + \cdots + w_q a_q \mid w_i \in \mathbf{R}\} \\ &= \mathrm{span}(\{a_1, a_2, \ldots, a_q\}) \end{aligned}$$

where $A = \left[\, a_1 \ \cdots \ a_q \,\right]$

$$\begin{aligned} \mathrm{nullspace}(B) &= \{x \mid Bx = 0\} \\ &= \{x \mid b_1^T x = 0, \ldots, b_p^T x = 0\} \end{aligned}$$

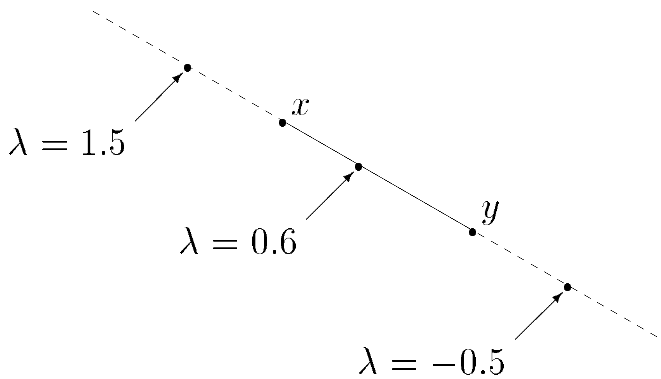
where $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_p^T \end{bmatrix}$

Affine sets

$S \subseteq \mathbf{R}^n$ is *affine* if

$$x, y \in S, \quad \lambda, \mu \in \mathbf{R}, \quad \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

Geometrically: $x, y \in S \Rightarrow$ line through $x, y \subseteq S$



Representations
range of affine function

$$S = \{Az + b \mid z \in \mathbf{R}^q\}$$

via linear equalities

$$S = \{x \mid b_1^T x = c_1, \ldots, b_p^T x = c_p\}$$

Convex sets

$S \subseteq \mathbf{R}^n$ is a *convex set* if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \lambda + \mu = 1 \Longrightarrow \lambda x + \mu y \in S$$

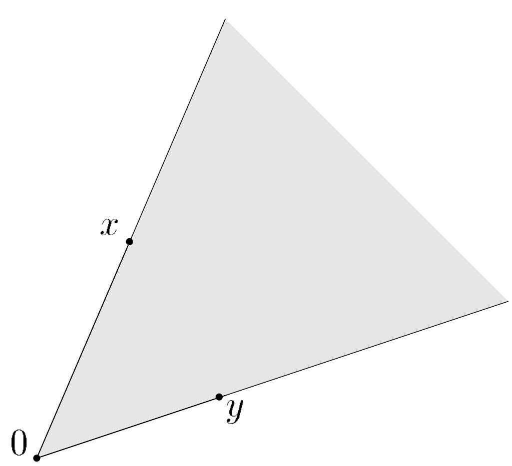
Geometrically: $x, y \in S \Rightarrow$ segment $[x, y] \subseteq S$

...many representations

$S \subseteq \mathbf{R}^n$ is a *convex cone* if

$$x, y \in S, \quad \lambda, \mu \geq 0, \quad \Longrightarrow \lambda x + \mu y \in S$$

Geometrically:
$x, y \in S \Rightarrow$ 2-dim. 'pie slice' between $x, y \subseteq S$



...many representations

Hyperplanes and halfspaces

Hyperplane $\{x \mid a^T x = b\}$ $(a \neq 0)$

affine; subspace if $b = 0$

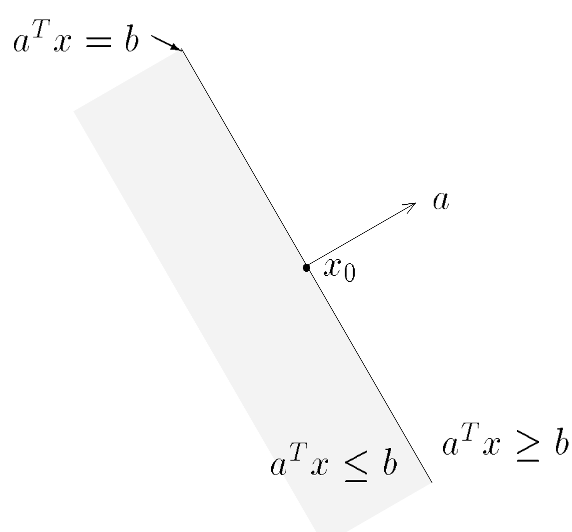
useful representation: $\{x \mid a^T(x - x_0) = 0\}$

a is *normal vector*; x_0 lies on hyperplane

Halfspace $\{x \mid a^T x \leq b\}$ $(a \neq 0)$

convex; convex cone if $b = 0$

useful representation: $\{x \mid a^T(x - x_0) \leq 0\}$
a is *(outward) normal vector*; x_0 lies on boundary



Intersections

$$S_\alpha \text{ is } \left(\begin{array}{l}\textbf{subspace}\\ \textbf{affine}\\ \textbf{convex}\\ \textbf{convex cone}\end{array}\right) \text{ for } \alpha \in \mathcal{A} \Longrightarrow \bigcap_{\alpha\in\mathcal{A}} S_\alpha \text{ is } \left(\begin{array}{l}\textbf{subspace}\\ \textbf{affine}\\ \textbf{convex}\\ \textbf{convex cone}\end{array}\right)$$

Example: *polyhedron* is intersection of finite number of halfspaces

$$\begin{aligned}\mathcal{P} &= \{x \mid a_i^T x \leq b_i,\ \ i = 1, \ldots, k\}\\ &= \{x \mid Ax \preceq b\}\end{aligned}$$

($\preceq$ means componentwise)

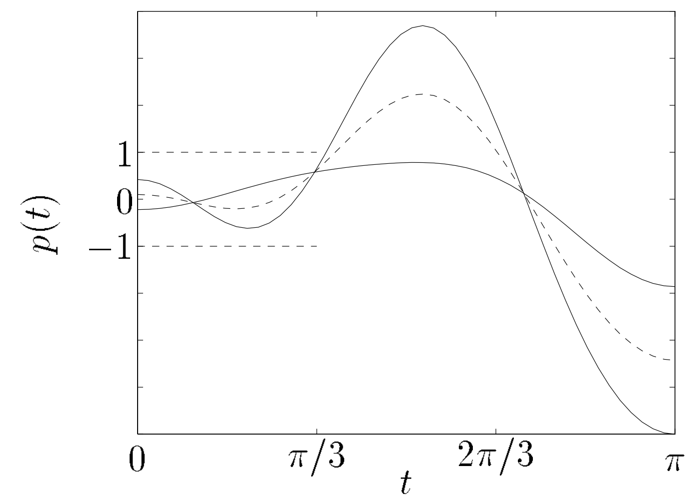
a bounded polyhedron is called a *polytope*

In fact, *every* closed convex set S is (usually infinite) intersection of halfspaces:

$$S = \cap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace},\ S \subseteq \mathcal{H}\}$$

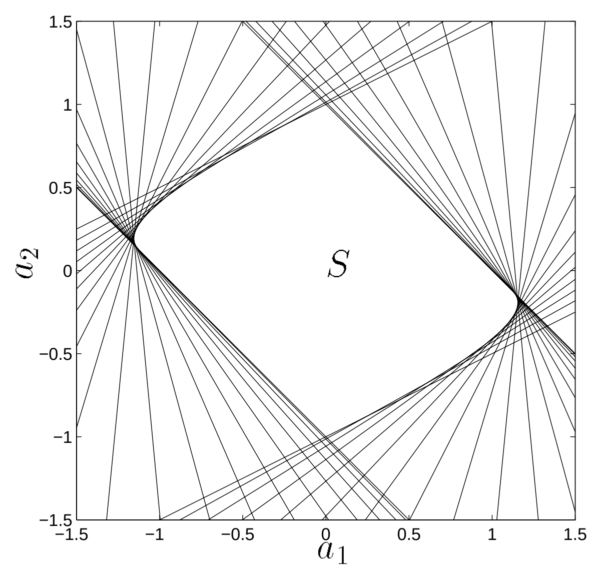
(more later)

Example: $S = \{a \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$, $p(t) = \sum_{k=1}^{m} a_k \cos kt$.



can express S as intersection of slabs: $S = \bigcap_{|t|\leq\pi/3} S_t$,

$$S_t = \{a \mid -1 \leq \left[\cos t \ \cdots \ \cos mt\right] a \leq 1\}.$$



Combinations and hulls

$y = \lambda_1 x_1 + \cdots + \lambda_k x_k$ is a

- *linear combination* of $x_1, \ldots, x_k$;
- *affine combination* if $\sum_i \lambda_i = 1$;
- *convex combination* if $\sum_i \lambda_i = 1, \ \lambda_i \geq 0$;
- *conic combination* if $\lambda_i \geq 0$.

(Linear,...) **hull** of S:
set of all (linear, ...) combinations from S

linear hull:	$\mathrm{span}(S)$
affine hull:	$\mathbf{Aff}(S)$
convex hull:	$\mathbf{Co}(S)$
conic hull:	$\mathbf{Cone}(S)$

$$\mathbf{Co}(S) = \cap\{G \mid S \subseteq G, \ G \text{ convex }\}, \ldots$$

Example

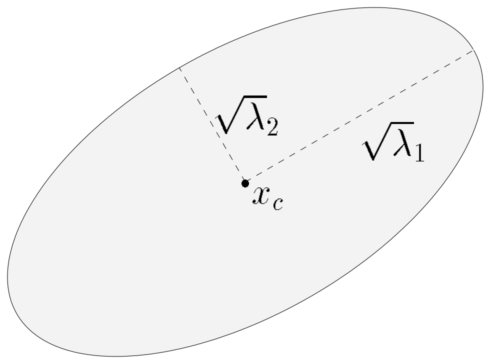
$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

what is linear, affine, ..., hull?

Ellipsoids

$$\mathcal{E} = \{x \mid (x - x_c)^T A^{-1}(x - x_c) \le 1\}$$

($A = A^T \succ 0$; $x_c \in \mathbf{R}^n$ center)



- semiaxis lengths: $\sqrt{\lambda_i}$; λ_i eigenvalues of A
- volume: $\alpha_n \left(\prod \lambda_i\right)^{1/2} = \alpha_n \left(\det A\right)^{1/2}$

Other descriptions

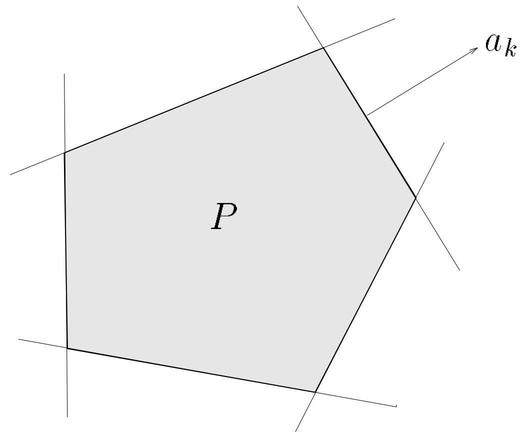
- $\mathcal{E} = \{Bu + x_c \mid \|u\| \le 1\}$ ($\|u\| = \sqrt{u^T u}$)
- $\mathcal{E} = \{x \mid f(x) \le 0\}$

$$\begin{aligned} f(x) &= x^T C x + 2d^T x + e \\ &= \begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} C & d \\ d^T & e \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \end{aligned}$$

($C = C^T \succ 0$, $e - d^T C^{-1} d < 0$)

Exercise: convert among representations; give center, semiaxes, volume.

Polyhedra



Examples

- nonnegative orthant $\{x \in \mathbf{R}^n \mid x \succeq 0\}$

- *k-simplex* $\mathbf{Co}\{x_0, \ldots, x_k\}$ with x_0, …, x_k affinely independent, *i.e.*,

$$\textbf{rank}\left(\begin{bmatrix} x_0 & x_1 & \cdots & x_k \\ 1 & 1 & \cdots & 1 \end{bmatrix}\right) = k+1,$$

  or equivalently, $x_1 - x_0, \ldots, x_k - x_0$ lin. indep.

- *standard simplex* $\{x \in \mathbf{R}^n \mid x \succeq 0, \ \sum_i x_i = 1\}$ also called *probability simplex*

Norm balls

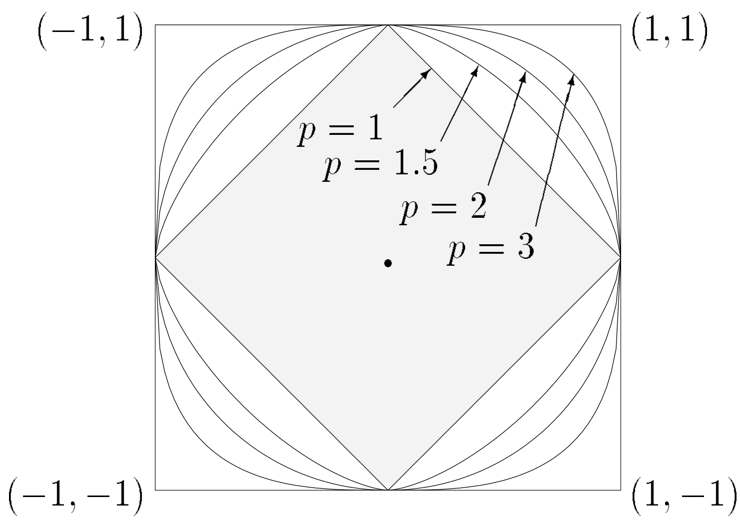
$f : \mathbf{R}^n \to \mathbf{R}$ is a *norm* if

1. $f(x) \geq 0$, $f(x) = 0 \Longrightarrow x = 0$
2. $f(tx) = |t| f(x)$, for all t
3. $f(x + y) \leq f(x) + f(y)$

(2),(3) $\Rightarrow$ the *ball* $\{x \mid f(x - x_c) \leq 1\}$ is convex.

Examples

- on $\mathbf{R}^n$: $\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$ $(p \geq 1)$;
  $\|x\|_\infty = \max_i |x_i|$



- on $\mathbf{R}^{m \times n}$: spectral norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sqrt{\lambda_{\max}(A^T A)}$$

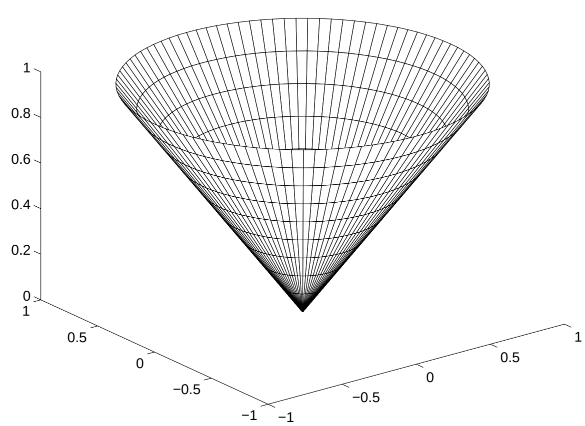
If $f(x)$ is a norm then

$$S = \{(x,t) \mid f(x) \le t\}$$

is a convex cone.

$e.g.$, Euclidean norm: the *second-order cone*, also called *quadratic* or *Lorentz* cone

$$\begin{aligned} S &= \{(x,t) \mid \sqrt{x^T x} \le t\} \\ &= \left\{ (x,t) \;\middle|\; \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0,\ t \ge 0 \right\} \end{aligned}$$



Affine transformations

suppose f is affine, *i.e.*, linear plus constant:

$$f(x) = Ax + b$$

if S, T convex, then so are

$$\begin{aligned} f^{-1}(S) &= \{x \mid Ax + b \in S\} \\ f(T) &= \{Ax + b \mid x \in T\} \end{aligned}$$

Example: coordinate projection

$$\left\{ x \,\middle|\, \begin{bmatrix} x \\ y \end{bmatrix} \in S \text{ for some } y \right\}$$

Linear matrix inequalities

$$\mathcal{P} = \{A \in \mathbf{R}^{n \times n} \mid A = A^T, A \succeq 0\}$$

is a convex cone, called the *positive semidefinite (PSD) cone*. ($A \succeq 0$ means positive semidefinite.)

$$\mathcal{P} = \bigcap_{z \in \mathbf{R}^n} \left\{ A = A^T \;\middle|\; z^T A z = \sum_{i,j} z_i z_j A_{ij} \geq 0 \right\},$$

i.e., intersection of infinite number of halfspaces in $\mathbf{R}^{n \times n}$

Hence, if A_0, A_1, ..., A_m symmetric, the solution set of the *linear matrix inequality*

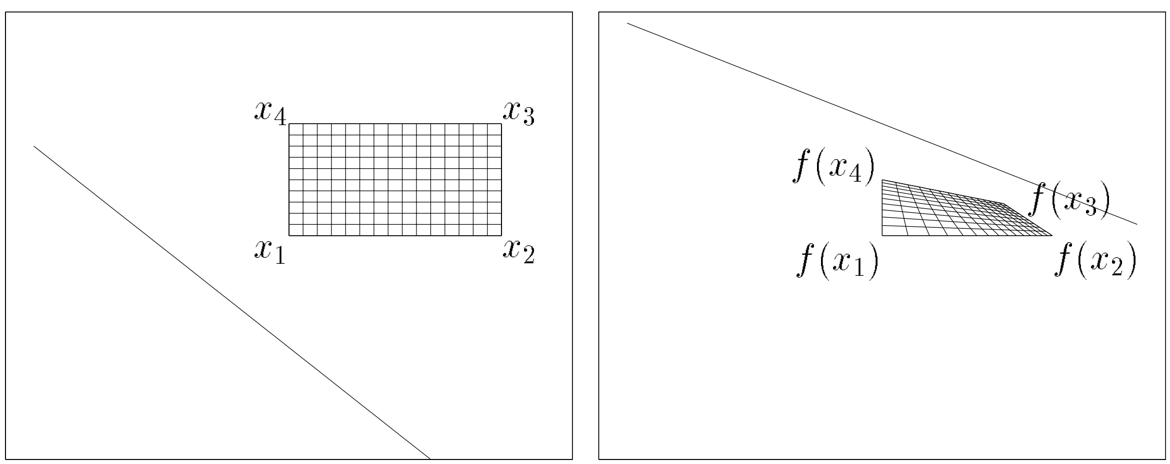
$$A_0 + x_1 A_1 + \cdots + x_m A_m \succeq 0$$

is convex

Projective transformation

$f : \mathcal{H} \to \mathbf{R}^n$, $\mathcal{H} = \{x \mid c^T x + d > 0\}$

$$f(x) = \frac{Ax + b}{c^T x + d}$$



Line segments preserved: for x, $y \in \mathcal{H}$,

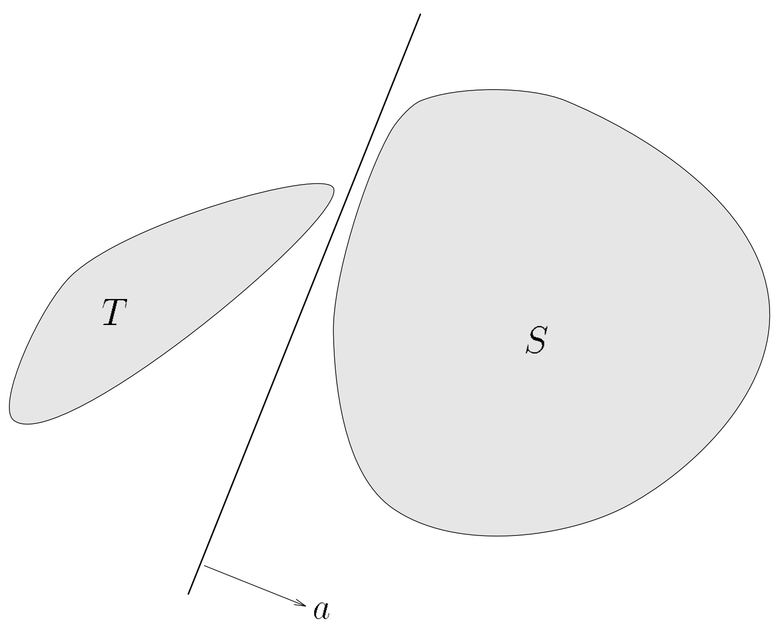
$$f([x, y]) = [f(x), f(y)]$$

Hence, if C convex, $C \subseteq \mathcal{H}$, then $f(C)$ convex.

Separating hyperplanes

$$S, T \text{ convex}, S \cap T = \emptyset$$
$$\Rightarrow \ \exists a \neq 0, b : \begin{cases} x \in S \Rightarrow a^T x \geq b \\ x \in T \Rightarrow a^T x \leq b \end{cases}$$

i.e., hyperplane $\{x \mid a^T x - b = 0\}$ *separates* S, T

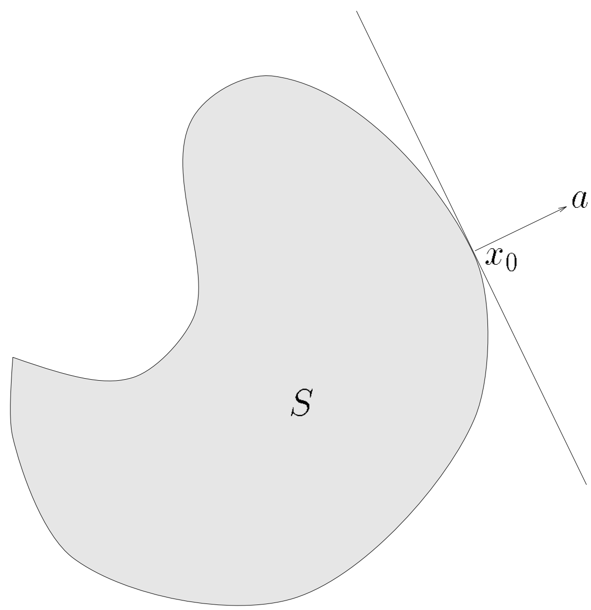


stronger forms use strict inequality, require conditions on S, T

Supporting hyperplane

Hyperplane $\{x \mid a^T x = a^T x_0\}$ ***supports*** S at $x_0 \in \partial S$ if

$$x \in S \Rightarrow a^T x \leq a^T x_0$$



halfspace $\{x \mid a^T x \leq b\}$ contains S for $b = a^T x_0$ but not for smaller b

S convex $\Rightarrow \exists$ supporting hyperplane for each $x_0 \in \partial S$

If S closed, $\textbf{int}\, S \neq \emptyset$, then

S convex $\Leftarrow \exists$ supporting hyperplane for each $x_0 \in \partial S$

Generalized inequalities

suppose convex cone $K \subseteq \mathbf{R}^n$

- is closed
- has nonempty interior
- is *pointed*: there is no line in K

K defines *generalized inequality* $\preceq_K$ in $\mathbf{R}^n$:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K$$

strict version:

$$x \prec_K y \quad \Longleftrightarrow \quad y - x \in \mathbf{int}\, K$$

examples:

- $K = \mathbf{R}^n_+$: $x \preceq_K y$ means $x_i \leq y_i$ (componentwise vector inequality)
- K is PSD cone in $\{X \in \mathbf{R}^{n\times n} | X = X^T\}$: $X \preceq_K Y$ means $Y - X$ is PSD

(these are so common we drop K)

many properties of $\preceq_K$ similar to $\leq$ on $\mathbf{R}$, *e.g.*,

- $x \preceq_K y,\ u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$
- $x \preceq_K y,\ y \preceq_K x \quad \Longrightarrow \quad x = y$

unlike $\leq$, $\preceq_K$ is not in general a *linear ordering*

Dual cones and inequalities

if K is a cone, *dual cone* is defined as

$$K^\star = \{\ y \mid x^T y \geq 0 \text{ for all } x \in K\ \}$$

for $K = \mathbf{R}^n_+$, $K^\star = K$, since

$$\sum_i x_i y_i \geq 0 \text{ for all } x_i \geq 0 \quad \Longleftrightarrow \quad y_i \geq 0$$

for $K =$ PSD cone, $K^\star = K$
(called *self-dual* cones)