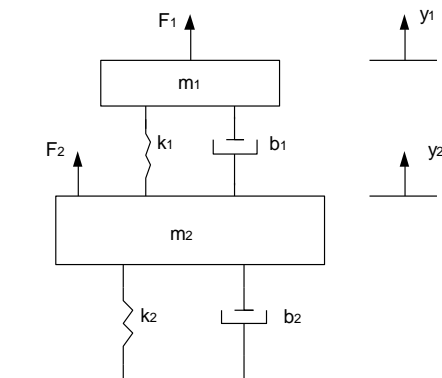


## ELEC-E8116 Model-based control systems /exercises and solutions 3

**Problem 1.** Consider the mass/spring/damper system shown in the figure. The control forces are  $F_1$  and  $F_2$ . Parameter values:  $k_1=1$ ,  $k_2=4$ ,  $b_1=0.2$ ,  $b_2=0.1$ ,  $m_1=1$ ,  $m_2=2$ .

- Form a differential model of the system.
- Form a state-space representation of the system.
- Plot the singular values as functions of frequency (Matlab).
- Calculate the  $H_\infty$ -norm (Matlab).



**Solution:**

- By applying Newton's 2. law

$$m_1 \ddot{y}_1 = F_1 - k_1(y_1 - y_2) - b_1(\dot{y}_1 - \dot{y}_2)$$

$$m_2 \ddot{y}_2 = F_2 + k_1(y_1 - y_2) + b_1(\dot{y}_1 - \dot{y}_2) - k_2 y_2 - b_2 \dot{y}_2$$

- Choose the state variables  $x_1 = y_1$ ,  $x_2 = y_2$ ,  $x_3 = \dot{y}_1$ ,  $x_4 = \dot{y}_2$  so that

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

$$m_1 \dot{x}_3 = F_1 - k_1(x_1 - x_2) - b_1(x_3 - x_4)$$

$$m_2 \dot{x}_4 = F_2 + k_1(x_1 - x_2) + b_1(x_3 - x_4) - k_2 x_2 - b_2 x_4$$

The state-space representation is

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

in which

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_1}{m_1} & \frac{k_1}{m_1} & \frac{-b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & \frac{-(k_1+k_2)}{m_2} & \frac{b_1}{m_2} & \frac{-(b_1+b_2)}{m_2} \end{bmatrix}$$

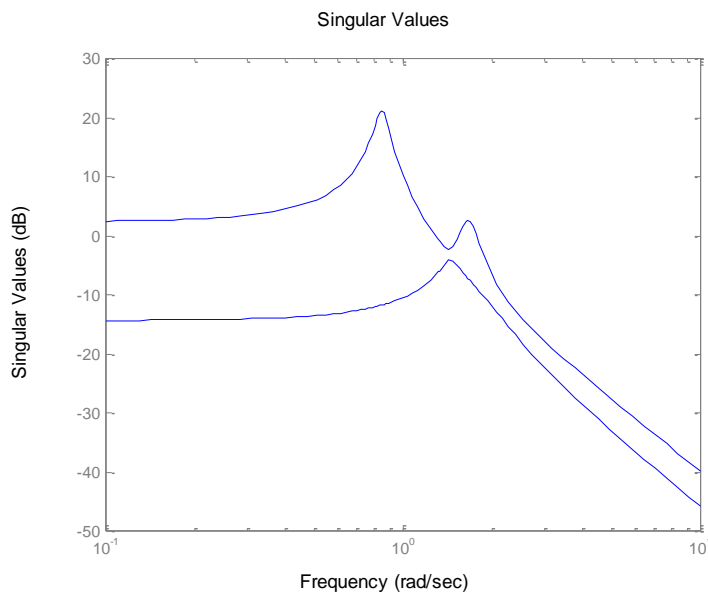
$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

When using Matlab also the  $D$  matrix must be given (with correct dimensions).

- c. In Matlab the command  $\text{sigma}(A,B,C,D)$  gives the figure



- d. The calculation of the  $H_\infty$ -norm of the transfer function is a difficult task (and no theoretical background for it has been given during the course either). Use the commands of the Robust Control Toolbox in Matlab (it is related to robust control theory, but the toolbox can be utilized in other purposes also).

1.  $G = \text{ss}(A,B,C,D)$ ;
2.  $\text{hinfnorm}(G,0.001)$ ; - calculate an approximation to the norm; tolerance (accuracy) 0.001. If desired, the tolerance is not needed.

The result is 11.4704. Note that in the figure the singular values are in dB-units. If you take the maximum of the larger singular value (approximately from the figure) and change it from logarithms to ordinary value you get pretty near 11.5 or so.

**Problem 2.** (Matlab) Consider the transfer function matrix

$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2 + 0.2s + 100} & \frac{1}{s+1} \\ \frac{s+2}{s^2 + 0.1s + 10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

Determine a realization and plot the singular values.

**Solution:**

Calculate the transfer function matrix by using Matlab

```
g11=10*tf([1 1],[1 0.2 100]);
g12=tf(1,[1 1]);
g21=tf([1 2],[1 0.1 10]);
g22=5*tf([1 1],conv([1 2],[1 3]));
G=[g11 g12;g21 g22];
```

Change into the state-space form

```
Gss=ss(G);
```

Form the *minimal realization* i.e. the representation with a minimal number of state variables that generates the same input-output behaviour.

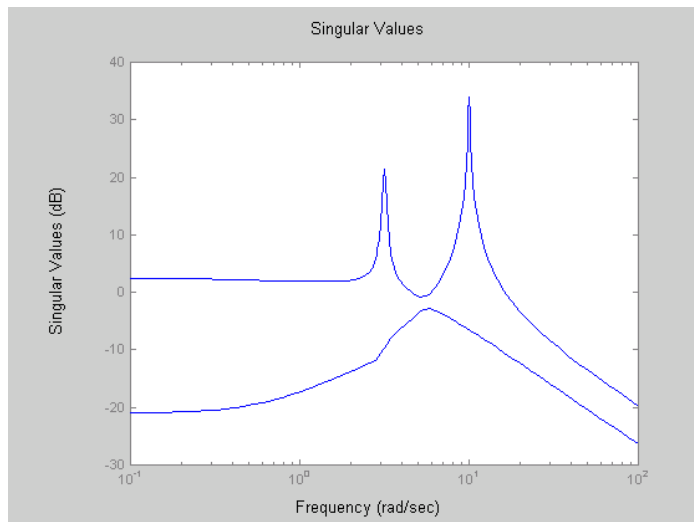
```
Gssm=minreal(ss(G));
```

Look at the matrices of the representation

```
[A,B,C,D]=ssdata(Gss);
[Am,Bm,Cm,Dm]=ssdata(Gssm);
```

We note that the first state-space representation was in fact the minimal realization ( $A = A_m$  etc.)

Plot the largest and smallest singular value  
 $\sigma(A,B,C,D)$



**Problem 3.** Two systems are given by the two transfer function matrices below. Calculate the poles and zeros

$$G_1(s) = \begin{bmatrix} \frac{2(s+1)(s+2)}{s(s+3)(s+4)} & \frac{s+2}{(s+1)(s+3)} \end{bmatrix}$$

$$G_2(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{s+3}{(s+1)(s-2)} \\ \frac{10}{(s-2)} & \frac{5}{s+3} \end{bmatrix}$$

**Solution:**

System 1: Minors  $\frac{2(s+1)(s+2)}{s(s+3)(s+4)}$ ,  $\frac{s+2}{(s+1)(s+3)}$

The pole polynomial  $p(s) = s(s+1)(s+3)(s+4)$ , from which the poles  $s = 0, s = -1, s = -3, s = -4$ . (All poles are single, so that the minimal realization has four states).

The “largest” minors (here the only ones have the degree 1. Arrange the pole polynomial to the denominator

$$\frac{2(s+1)^2(s+2)}{s(s+1)(s+3)(s+4)}, \quad \frac{s(s+2)(s+4)}{s(s+1)(s+3)(s+4)}$$

The zero polynomial is  $z(s) = s+2$ , so that the zero is  $s = -2$ .  
But what about Matlab. Let us try

```
G1=tf([2 6 4],[1 7 12 0]),tf([1 2],[1 4 3]);
pole(G1)
```

ans =

```
0
-4
-3
-3
-1
```

One pole appears twice. Try the minimal realization

```
pole(minreal(G1))
```

ans =

```
0
-4
-3
-3
-1
```

Same results. But programs are only programs! Form first a state-space realization, and from it the minimal realization

```
G1ss=minreal(ss(G1))
1 state(s) removed.
```

Now the extra state disappeared

a =

	x1	x2	x3	x4
x1	-3.0047	0.33629	-0.35451	-0.57787
x2	2.6142	-0.31631	-0.18814	0.39683
x3	2.4213	-0.47446	-4.2822	-0.90475
x4	-1.6142	0.31631	2.1881	-0.39683

b =

	u1	u2
x1	-1.4021	0.16105
x2	0.46736	-0.053685
x3	0.70105	0.91947
x4	-0.46736	0.053685

c =

	x1	x2	x3	x4
y1	-1.2095	-0.34684	1.2297	0.84684

d =

$$\begin{array}{ccc} & u1 & u2 \\ y1 & 0 & 0 \end{array}$$

Continuous-time model.

pole(G1ss)

ans =

-3.0000  
-4.0000  
-0.0000  
-1.0000

The result is correct. The Matlab command  $tzero(G_1)$  (transmission zero) gives the zero  $-2$ , as desired.

System 2: Minors

$$\frac{1}{s+1}, \frac{s+3}{(s+1)(s-2)}, \frac{10}{s-2}, \frac{5}{s+3},$$
$$\frac{5}{(s+1)(s+3)} - \frac{10(s+3)}{(s+1)(s-2)^2} = \dots = \frac{-5(s^2 + 16s + 14)}{(s+1)(s+3)(s-2)^2}$$

Pole polynomial  $p(s) = (s+1)(s-2)^2(s+3)$ , poles  $s = -1, 2, -3$ .

The pole 2 has the degree 2; the minimal realization has four states.

Zeros: the largest minor has the dimension 2. Arrange again the pole polynomial to the denominator

$$z(s) = \frac{-5(s^2 + 16s + 14)}{(s+1)(s-2)^2(s+3)}$$

Zeros  $-15.07, -0.93$  (both have the degree one). Confirm with Matlab.

**Problem 4.** By considering the static system

$$G(s) = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}$$

prove that eigenvalues do not give a reliable view about the gain of a multivariable system. What is a better alternative?

**Solution:** The eigenvalues of the matrix are both zero. However, the input  $u = [0 \ 1]^T$  gives the output  $y = [100 \ 0]^T$ . So, at least in this input direction the eigenvalue 0 does not describe the system well. The problem is that generally eigenvalues describe the gain only in the direction of the eigenvector. Let  $(\lambda_i, t_i)$  be the eigenvalue-eigenvector pair related to the matrix  $G$ . For the eigenvector as input

$$y = Gt_i = \lambda_i t_i \quad \Rightarrow \quad \frac{|y|}{|t_i|} = \frac{|\lambda_i t_i|}{|t_i|} = |\lambda_i|$$

But let us calculate the singular values.

By the Matlab command  $svd(G)$  or by calculating directly  $\text{sqrt}(\text{eig}(G^*G))$  the singular values 100 and 0 are obtained. The gain of the system is between these values, and the maximum gain (corresponding to the infinity norm of the transfer function) is 100.

Note. The singular values are obtained as  $\sigma_i(G) = \sqrt{\lambda_i(G^H G)}$ ; the eigenvalues here are always real, because  $G^H G$  is a hermitian matrix. Also  $G^H G$  is positive semidefinite, so that the eigenvalues are nonnegative.