ELEC-E8116 Model-based control systems
/exercises with solutions 2

Problem 1

Prove that the solution of the state equation $\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0$ is

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Solution 1

First some information, which is needed. The matrix exponential is defined as a series

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$$

which always converges. There exists at least 19 ways to determine this function analytically (see. Moler, C., Van Loan, C.: "Nineteen dubious ways to compute the exponential of a matrix", *SIAM Review*, Vol. 20, pp. 801-836, 1978). One way is well-known

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$$

in which the inverse Laplace transformation is sometimes difficult. In practice, if one method turns out to be difficult in some example case, the other methods are hardly easier.

The rule for derivation follows easily

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{1}{2!}A^3t^2 + \cdots = A(I + At + \frac{1}{2!}(At)^2 + \cdots = Ae^{At}$$

which makes sense, of course (but in matrix calculus you can never take for granted that the familiar rules for scalar systems inevitably hold).

Remember the derivation rule

$$\frac{d}{dt}\int_{u(t)}^{v(t)} f(x,t)dx = -f(u(t),t)\dot{u}(t) + f(v(t),t)\dot{v}(t) + \int_{u(t)}^{v(t)} \frac{\partial f(x,t)}{\partial t}\ dx$$

after which the problem becomes tractable. First, check the initial condition by substituting t_0 into the solution formula given in the problem

$$x(t_0) = Ix_0 + 0 = x_0$$

so the initial condition holds. Then let us check, whether the candidate solution fulfills the differential equation

$$\begin{aligned}\dot{x}(t) &= Ae^{A(t-t_0)}x_0 - 0 + e^{A(t-t)}Bu(t)\cdot 1 + \int_{t_0}^{t} Ae^{A(t-\tau)}Bu(\tau)\,d\tau \\ &= Ae^{A(t-t_0)}x_0 + Bu(t) + A\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \\ &= Ax(t) + Bu(t)\end{aligned}$$

Yes it does.

Problem 2

Derive the general solution for the discrete-time difference equation

$$x(t+1) = Ax(t) + Bu(t), \quad x(t_0) = x_0$$

Solution 2

Discrete-time equations are solution algorithms as such. Calculate directly

$$\begin{aligned}
&x(t_0+1) = Ax_0 + Bu(t_0) \\
&x(t_0+2) = Ax(t_0+1) + Bu(t_0+1) = A^2x_0 + ABu(t_0) + Bu(t_0+1) \\
&x(t_0+3) = Ax(t_0+2) + Bu(t_0+2) = A^3x_0 + A^2Bu(t_0) + ABu(t_0+1) + Bu(t_0+2) \\
&\vdots \\
&x(t_0+N) = A^N x_0 + A^{N-1}Bu(t_0) + \cdots + ABu(t_0+N-2) + Bu(t_0+N-1)
\end{aligned}$$

which gives

$$\begin{aligned}
x(t) &= A^{t-t_0}x_0 + A^{t-t_0-1}Bu(t_0) + \cdots + ABu(t-2) + Bu(t-1) \\
&= A^{t-t_0}x_0 + \sum_{i=t_0}^{t-1} A^{t-i-1}Bu(i)
\end{aligned}$$

Problem 3

The *trace* of a square matrix is defined as the sum of the elements in the main diagonal. Let *A* and *B* be square matrices of equal dimensions and *C* and *D* such matrices that *CD* and *DC* are both properly defined. Prove

a. $tr(A+B) = tr(A)+tr(B)$
b. $tr(CD) = tr(DC)$

Solution 3

Let the dimensions of *A*, *B*, *C* and *D* be *nxn*, *nxn*, *nxm* and *mxn*. Use the abbreviation (a_{ij}) for the elements of the matrices. It follows

$$tr(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n}a_{ii} + \sum_{i=1}^{n}b_{ii} = tr(A)+tr(B) \qquad \text{Ok.}$$

$$CD(i,j) = \sum_{k=1}^{m} c_{ik}d_{kj}$$ (the component *i,j* of CD by the definition of matrix multiplication)

By setting $i = j$ and calculating the sum of the components (the matrix trace)

$$tr(CD) = \sum_{i=1}^{n}\sum_{k=1}^{m} c_{ik}d_{ki}$$

Correspondingly

$$DC(j,i) = \sum_{k=1}^{n} d_{jk} c_{ki}$$

$$tr(DC) = \sum_{i=1}^{m}\sum_{k=1}^{n} d_{ik} c_{ki} \;= \sum_{i=1}^{n}\sum_{k=1}^{m} c_{ik} d_{ki}$$

The results are the same, Ok.

Note: Sometimes the same result is presented in the following form - let A and B be nxm-matrices. Then it holds

$$tr(AB^T) = tr(A^T B)$$

This is easily proved by the previous result and by noting that in taking the transpose of a square matrix the elements in the main diagonal remain the same (trace is constant). Hence

$$tr(AB^T) = tr\left[\left(AB^T\right)^T\right] = tr(BA^T) = tr(A^T B)$$

Problem 4

Square matrices B and A of equal dimensions are called *similar*, if there exists an invertible square matrix T such that

$$B = TAT^{-1}$$

Prove that similar matrices have the same eigenvalues, the same *trace* and the same determinant.

Solution 4

Let us first show a result: For any invertible matrix T it holds

$$\det(T^{-1}) = \frac{1}{\det(T)}$$

It is known that for square matrices A and B of equal dimensions it holds

$$\det(AB) = \det(A)\det(B)$$

By applying that to the identity

$$TT^{-1} = I$$

it follows

$$\det(TT^{-1}) = \det(T)\det(T^{-1}) = 1$$

as claimed.

Then to the problem: form the characteristic polynomial of B

$$\begin{aligned}\det(\lambda I - B) &= \det(\lambda I - TAT^{-1}) = \det(\lambda TT^{-1} - TAT^{-1}) \\ &= \det(T(\lambda I - A)T^{-1}) = \det(T)\det(\lambda I - A)\det(T^{-1}) \\ &= \underbrace{\det(T)\det(T^{-1})}_{1}\det(\lambda I - A) = \det(\lambda I - A)\end{aligned}$$

The characteristic polynomials are the same and therefore the eigenvalues are also the same.

What about trace? Apply directly the result given in problem 8 b; choose $C = TA$ and $D = T^{-1}$. We obtain

$$tr(B) = tr(TA \cdot T^{-1}) = tr(T^{-1}TA) = tr(IA) = tr(A)$$

For the determinant

$$\det(B) = \det(TAT^{-1}) = \det(T)\cdot\det(A)\cdot\det(T^{-1}) = \det(T)\cdot\det(A)\cdot\frac{1}{\det(T)} = \det(A)$$

Problem 5

Consider the MISO-system

$$y(t) = \frac{p+2}{p^2+2p+1}u_1(t) + \frac{1}{p^2+3p+2}u_2(t)$$

Form a realization (state-space representation).

Solution 5

Apply the "systematic" method

$$y(t) = \frac{p+2}{(p+1)^2}u_1(t) + \frac{1}{(p+1)(p+2)}u_2(t) = \frac{(p+2)^2u_1(t) + (p+1)u_2(t)}{(p+1)^2(p+2)}$$

$$\left(p^3+4p^2+5p+2\right)y(t) = (p^2+4p+4)u_1(t) + (p+1)u_2(t)$$

$$p^3y + 4p^2y + 5py - p^2u_1 - 4pu_1 - pu_2 = -2y + 4u_1 + u_2$$

$$\underbrace{p\left\{\underbrace{p\left[p\underbrace{y}_{x_1} - u_1 + 4y\right]}_{x_2} + 5y - 4u_1 - u_2\right\}}_{x_3} = -2y + 4u_1 + u_2$$

It follows

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{x}_1 - u_1 + 4x_1 \\ x_3 &= \dot{x}_2 + 5x_1 - 4u_1 - u_2 \end{aligned}$$

and easily

$$\begin{aligned} \dot{x}_1 &= -4x_1 + x_2 + u_1 \\ \dot{x}_2 &= -5x_1 + x_3 + 4u_1 + u_2 \\ \dot{x}_3 &= -2x_1 + 4u_1 + u_2 \\ y &= x_1 \end{aligned}$$

which is in the *observable canonical form*.

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 \\ -5 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 4 & 1 \\ 4 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(see textbook pp. 35-37).

Problem 6

Consider the following optimization problem. Let

$$\dot{x}(t) = u(t), \quad x(0) = 1$$

and find an optimal control u, which minimizes the criterion

$$J = \int_0^\infty \left[x^2(t) + u^2(t) \right] dt$$

Prove that the solution can be presented in state feedback form $u^*(t) = -x(t)$. What is the optimal control as a function of time? What is the optimal cost?

Hint: Prove first the identity

$$\int_0^T \left[x^2(t) + \dot{x}^2(t) \right] dt = x^2(0) - x^2(T) + \int_0^T \left[x(t) + \dot{x}(t) \right]^2 dt$$

Solution 6

The identity follows easily by developing the "right hand side" and by noticing that

$$\frac{d}{dt}\left[x(t) \right]^2 = 2x(t)\dot{x}(t)$$

Because $\dot{x}(t) = u(t)$ (system dynamics), the criterion can be written as

$$J = \int_0^T \left(x^2 + u^2 \right) dt = \int_0^T \left(x^2 + \dot{x}^2 \right) dt = x^2(0) - x^2(T) + \int_0^T \left(x + u \right)^2 dt$$

which attains the minimum, when $u^*(t) = -x(t)$ (the final state is free, so there is no need to bother about boundary conditions).

The solution is valid for all T, also for an infinite optimization horizon.

By substituting the optimal control to the system equation leads to

$$\dot{x}(t) = -x(t), \quad x(0) = 1$$

which is solved easily e.g. by using the Laplace-transformation

$$sX(s) - x(0) = -X(s)$$
$$(s+1)X(s) = 1$$
$$X(s) = \frac{1}{s+1}$$

The optimal trajectory is

$$x^*(t) = e^{-t}$$

and the optimal cost is easily determined by direct calculus

$$J^* = \int_0^\infty \left(x^2(t) + u^2(t)\right)dt = \int_0^\infty \left(e^{-2t} + e^{-2t}\right)dt = 1$$