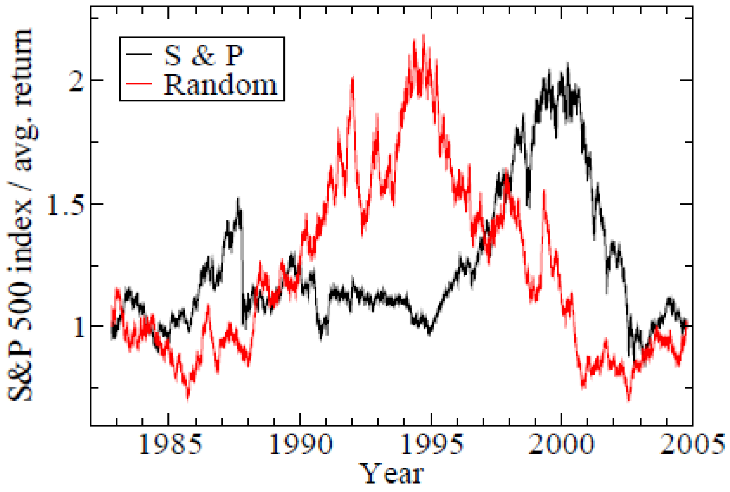
Statistical Mechanics
E0415

Fall 2023, lecture 2
Random walks

Universality, scale invariance

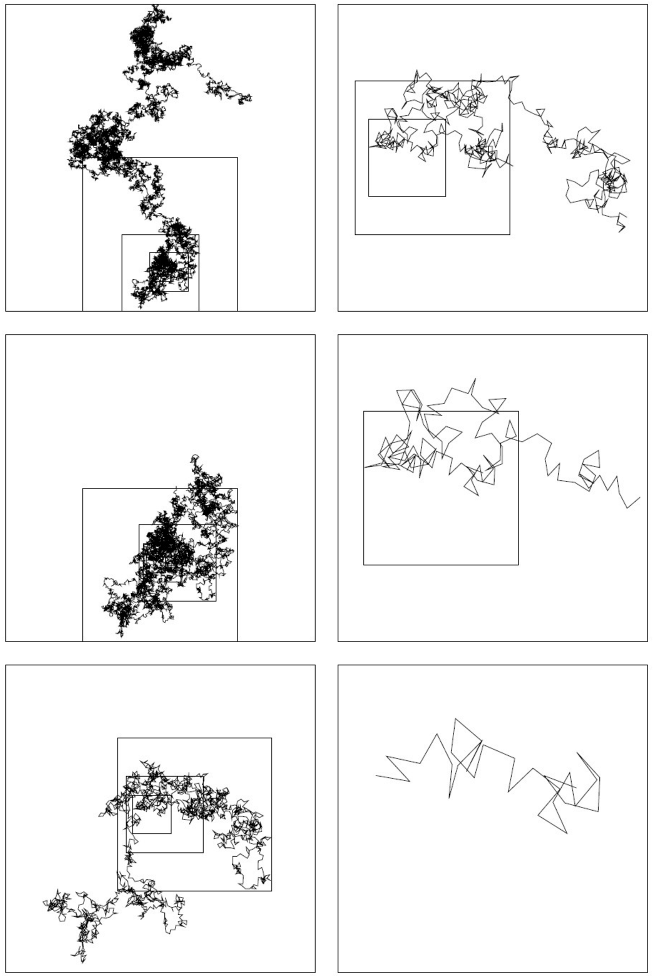
- **Three main points with random walks**: scaling (scale-free) behavior, universality (small details do not matter), probability distributions (and the governing equation(s)).
- Example: coin flips/tosses – do heads or tails win? Square-root law with N.
- Example II: drunkard’s walk (on a 2D plane).
- Universality – compare with polymers (self-avoidance). “Entropic repulsion”, walk exponent becomes ϒ = ¾ (2D), 0.59 (3) – exact and numerical values. “Universal critical exponent” (Self-Avoiding Walks, SAW).

Sorts of RWs



Stock Exchange Index vs.
a simple RW

Fig. 2.2 Random walk: scale invariance. Random walks form a jagged, fractal pattern which looks the same when rescaled. Here each succeeding walk is the first quarter of the previous walk, magnified by a factor of two; the shortest random walk is of length 31, the longest of length 32 000 steps. The left side of Fig. 1.1 is the further evolution of this walk to 128 000 steps.



The Diffusion Equation

An equation for ρ: two interpretations – density of a cloud, pdf of a single RW.

$$\frac{\partial \rho}{\partial t} = D \nabla^2 \rho = D \frac{\partial^2 \rho}{\partial x^2}.$$

Derivation of DE: separation of scales (RW step against the gradient of ρ).

Relation of the diffusion constant D > 0 to the microscopic RW details:

$$D = a^2 / 2\Delta t.$$

Step size a, timescale Δt.

Currents & external forces

Remember: DE conserves particles, thus the density – "conservation law".

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}. \qquad J_{\text{diffusion}} = -D\frac{\partial \rho}{\partial x},$$

In the presence of external forces the particles have a deterministic drift.

$$x(t+\Delta t) = x(t) + F\gamma\Delta t + \ell(t).$$

This shows up in the current, and in the equation for ρ.

$$J = \gamma F \rho - D\frac{\partial \rho}{\partial x}.$$

$$\frac{\partial \rho}{\partial t} = -\gamma F\frac{\partial \rho}{\partial x} + D\frac{\partial^2 \rho}{\partial x^2},$$

Case study: density profile with gravity ("atmosphere").

$$\rho^*(x) = A\exp\left(-\frac{\gamma}{D}mgx\right).$$

Solving the diffusion equation

Example: Fourier method. FT in space, substitute: reveals the diffusive timescale and the role of D.

General solution as superposition of the FT of the initial profile or condition .

$$\frac{\partial \rho}{\partial t} = \frac{\mathrm{d}\widetilde{\rho}_k}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}kx} = D\frac{\partial^2 \rho}{\partial x^2} = -Dk^2\widetilde{\rho}_k\mathrm{e}^{\mathrm{i}kx},$$

$$\frac{\mathrm{d}\widetilde{\rho}_k}{\mathrm{d}t} = -Dk^2\widetilde{\rho}_k,$$

$$\widetilde{\rho}_k(t) = \widetilde{\rho}_k(0)\mathrm{e}^{-Dk^2 t}.$$

$$\rho(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\rho}_k(0)\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-Dk^2 t}\,\mathrm{d}k.$$

$$\widetilde{\rho}_k(0) = \int_{-\infty}^{\infty}\rho(x,0)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

Homework

1.2 Waiting times (Sethna 1.3 p. 6) HOMEWORK (5 points)

Study Sethna Ch2, answer the following exercise in gambling. You play heads and tails (toss a coin, and guess the outcome: win or lose the coin).

Three questions: you start with 10 coins. Give an argument how the distribution of times it takes for you to lose all your coins looks like.

What happens if you play till you have zero, or until you won all the 10 coins of your friend?

Let us now consider the case where the coin is not fair: the fractional Brownian motion, where the subsequent outcomes are correlated (positively or negatively). How does that influence qualitatively those outcomes?

Other scheduling

Paper presentations and groups and project groups: DL tomorrow, we summarize the situation and make the groups and inform you (MC) early next week. First paper presentation probably 6th of October – so the group HAS to be present. If you can not be present at your slot: contact us ASAP.

Friday lectures: we usually do 13.15