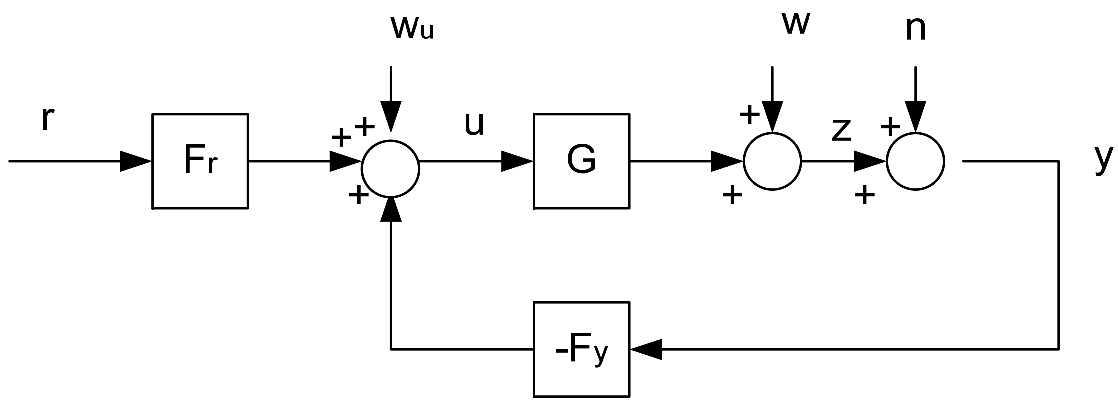
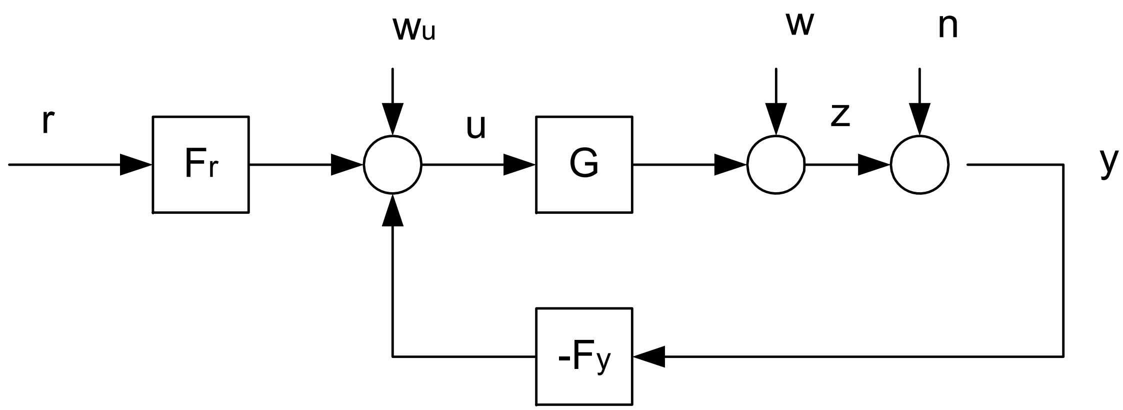
Closed-loop system



Two-degrees-of-freedom (2 DOF) control structure

$$u(t) = F_r(p)r(t) - F_y(p)y(t)$$

Generally MIMO case



$$z(t) = G(p)u(t) + w(t)$$

$$y(t) = z(t) + n(t)$$

$$\bar{u}(t) = F_r(p)r(t) - F_y(p)y(t)$$ (2 DOF structure)

$$u(t) = \bar{u}(t) + w_u$$

Aalto University
School of Electrical
Engineering

Earlier the **matrix inversion lemma** was presented.

Now consider the **”push-through”-rule**, which is often useful in matrix manipulations related to multivariable systems. In the transfer function matrix representations that follow the push-through rule is often used.

Let *A* and *B* be such matrices that both *AB* and *BA* are defined and square matrices. Then it holds

$$A(I + BA)^{-1} = (I + AB)^{-1} A$$

Note. *A* and *B* need not be square matrices. The matrix inverses above are assumed to exist.

The closed loop equations become

$$z = GF_r r - GF_y z - GF_y n + w + Gw_u$$

solving for *z* gives

$$z = (I + GF_y)^{-1} GF_r r + (I + GF_y)^{-1} w - (I + GF_y)^{-1} GF_y n$$
$$+ (I + GF_y)^{-1} Gw_u$$

Control error $e = r - z$

Use the following abbreviations

$$z = G_c r + Sw - Tn + GS_u w_u$$

$$e = (I - G_c)r - Sw + Tn - GS_u w_u$$

where G_c is the *closed loop transfer function (matrix)*

$$G_c = (I + GF_y)^{-1} GF_r$$

S is the *sensitivity function* $S = (I + GF_y)^{-1}$

T is the *complementary sensitivity function*

$$T = (I + GF_y)^{-1} GF_y$$

S_u is the input sensitivity function

$$S_u = (I + F_y G)^{-1}$$

Generally: $S(j\omega) + T(j\omega) = I$

(in all frequencies)

Fundamental relationship
The most important formula
in control engineering!

Note that the transfer functions (and matrices) are complex-valued.

Often $F_y = F_r$ in which case $T = G_c$

(One-degree-of-freedom (1 DOF) control configuration)

What about the control signal

$$u = (I + F_y G)^{-1} F_r r - (I + F_y G)^{-1} F_y (w + n) + (I + F_y G)^{-1} w_u$$
$$= S_u F_r r - S_u F_y (w + n) + S_u w_u = G_{ru} r + G_{wu} (w + n) + S_u w_u$$

where

$$G_{ru} = (I + F_y G)^{-1} F_r$$
$$G_{wu} = -(I + F_y G)^{-1} F_y$$

System with inputs r, w, w_u, n

and outputs z, e

Aalto University
School of Electrical Engineering

Note that the *loop transfer function*

$$L(j\omega) = G(j\omega)F_y(j\omega)$$ (Classical Bode analysis from that)

is obtained, when the controller has been designed. That implies also

$$S(j\omega) = \left[I + L(j\omega)\right]^{-1}$$

$$T(j\omega) = L(j\omega)\left[I + L(j\omega)\right]^{-1}$$

which charecterise the opetation of the loop. The closed loop transfer function includes also the pre-filter, which is outside the loop.

$$G_c = (I + GF_y)^{-1} GF_r$$

Designing F_y and for the servo problems also F_r

such that *L, S* and *T* are as desired is called
loop shaping.

Note that in classical control the design of compensators that lead to desired gain and phase margins is loop shaping also.

Now we are broadening the view, which leads to new tools for controller analysis and synthesis. New players: *S* and *T*.

Internal stability

Zero-pole-cancellations are problematic

Ex. $G(s) = \frac{s-1}{s+1}$ process

$F_y(s) = F_r(s) = \frac{1}{s-1}$ controller

$G_c = \frac{\frac{s-1}{s+1} \cdot \frac{1}{s-1}}{1 + \frac{s-1}{s+1} \cdot \frac{1}{s-1}} = \frac{1}{s+2}$ closed loop system; stable

But how about transfer function from reference to control?

$$G_{ru} = \frac{\dfrac{1}{s-1}}{1+\dfrac{s-1}{s+1}\cdot\dfrac{1}{s-1}} = \frac{s+1}{(s-1)(s+2)}$$

The controller is unstable, and therefore useless .

The unstable mode corresponding to s – 1 was not observable in the closed loop transfer function.

Ex. $G(s) = \frac{1}{s-1}$ process

$$F_y(s) = F_r(s) = \frac{s-1}{s+1}$$ controller

$$G_c = \frac{\frac{1}{s-1} \cdot \frac{s-1}{s+1}}{1 + \frac{1}{s-1} \cdot \frac{s-1}{s+1}} = \frac{1}{s+2}$$ closed loop

Closed loop from *r* (or *w*) to control *u*

$$G_{ru} = \frac{\frac{s-1}{s+1}}{1 + \frac{1}{s-1} \cdot \frac{s-1}{s+1}} = \frac{s-1}{s+2}$$

Sensitivity function

$$S = \frac{1}{1 + \frac{1}{s-1} \cdot \frac{s-1}{s+1}} = \frac{s+1}{s+2}$$

All stable; is there no *s* – 1 problem now?
Calculate the transfer function from w_u to output.

$$G_{w_u y}(s) = (I + F_y G)^{-1} G$$

or

$$G_{w_u y}(s) = \frac{\frac{1}{s-1}}{1 + \frac{1}{s-1} \cdot \frac{s-1}{s+1}} = \frac{s+1}{(s-1)(s+2)}$$

unstable!
r does not "activate" this mode

But do ”safe” cancellations exist?

Ex. $G(s) = \dfrac{1}{s-1}$ $\quad F_r = F_y = 3$ The process is unstable

$$G_c = \frac{\frac{3}{s-1}}{1+\frac{3}{s-1}} = \frac{3}{s+2}$$

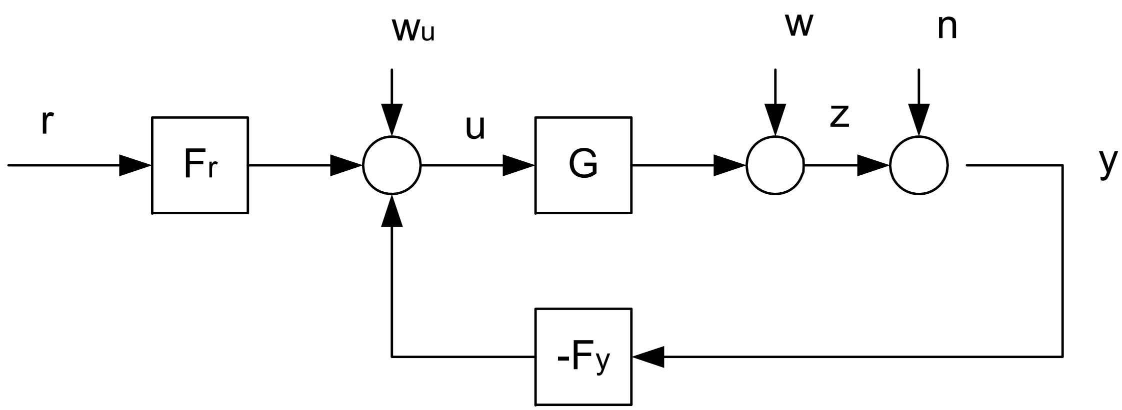
$$G_{ru} = \frac{3}{1+\frac{3}{s-1}} = \frac{3(s-1)}{s+2} \qquad S = \frac{1}{1+\frac{3}{s-1}} = \frac{s-1}{s+2}$$

$$G_{w_u y} = \frac{\frac{1}{s-1}}{1+\frac{3}{s-1}} = \frac{1}{s+2}$$

all is nice and beautiful; Feedback can stabilize an unstable process. Reason for a definition:

Aalto University
School of Electrical Engineering

Internal stability of the closed-loop system



The system is internally stable, if (after all cancellations in the calculation of the transfer functions have been made) the following transfer functions

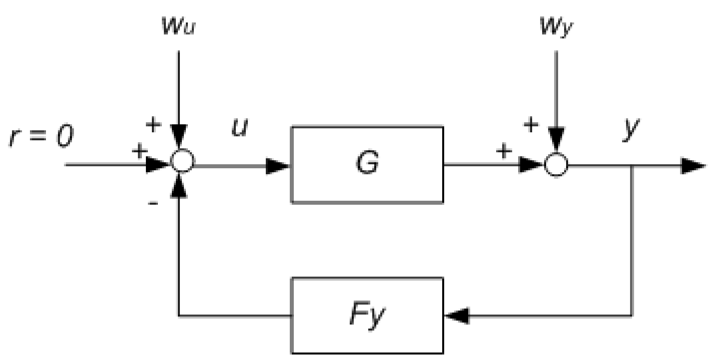
$$w_u \mapsto u, \qquad S_u = G_{w_u u} = (I + F_y G)^{-1}$$

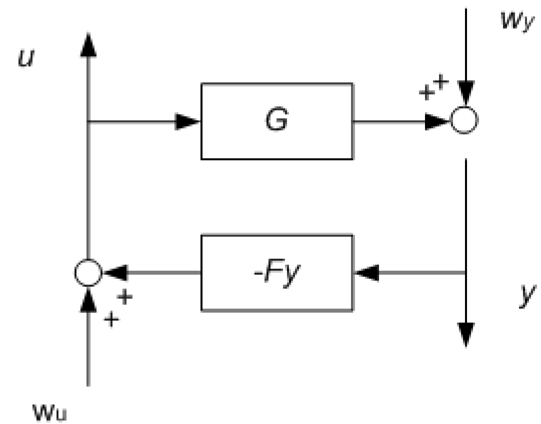
$$w_u \mapsto y, \qquad G_{w_u y} = (I + GF_y)^{-1} G$$

$$w_y \mapsto u, \qquad G_{w_y u} = -(I + F_y G)^{-1} F_y$$

$$w_y \mapsto y, \qquad S = G_{w_y y} = (I + GF_y)^{-1}$$

are stable and the pre-compensator *Fr* is also stable.





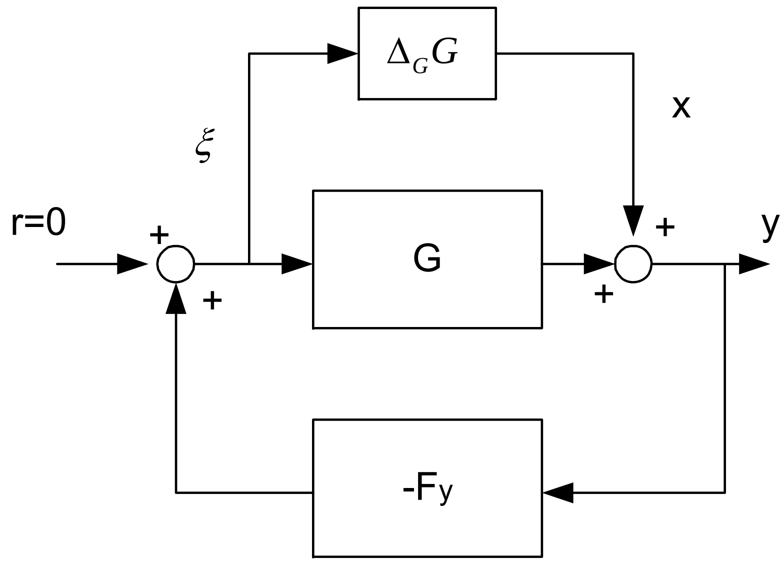
$$u = w_u - F_y(w_y + Gu) = w_u - F_y w_y - F_y G u$$

$$\Rightarrow u = \left(I + F_y G\right)^{-1} w_u - \left(I + F_y G\right)^{-1} F_y w_y$$

Similarly:

$$y = \left(I + GF_y\right)^{-1} w_y + \left(I + GF_y\right)^{-1} G w_u$$

Robustness



$$G_0 = (I + \Delta_G)G$$

nominal model G
true system G_0

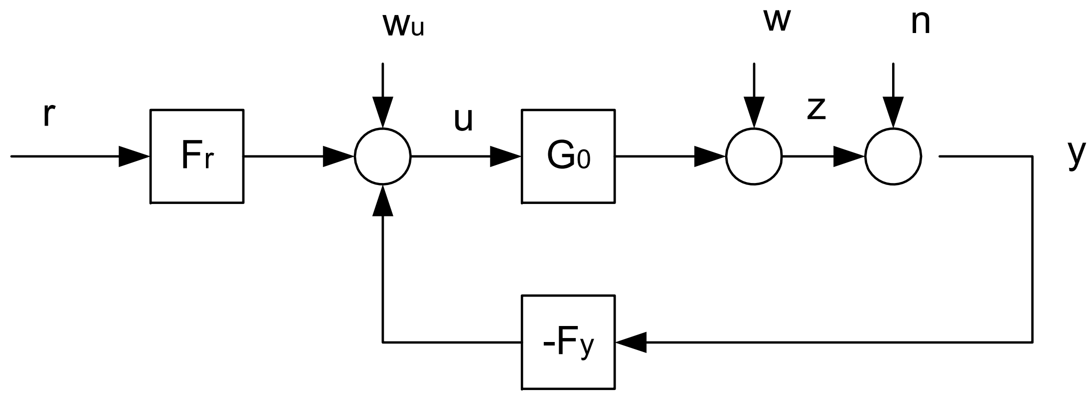
model error Δ_G

Is the closed loop stable in spite of model error?

(robust stability)

Does the system meet performance specifications in spite of model error ? (robust performance)

$$G_0 = (I + \Delta_G)G$$ nominal model *G*



Assume that the disturbances are zero, but there is model error in *G*

$$z_0 = (I + G_0 F_y)^{-1} G_0 F_r r$$ real output

$$z = (I + G F_y)^{-1} G F_r r$$ output predicted by the model

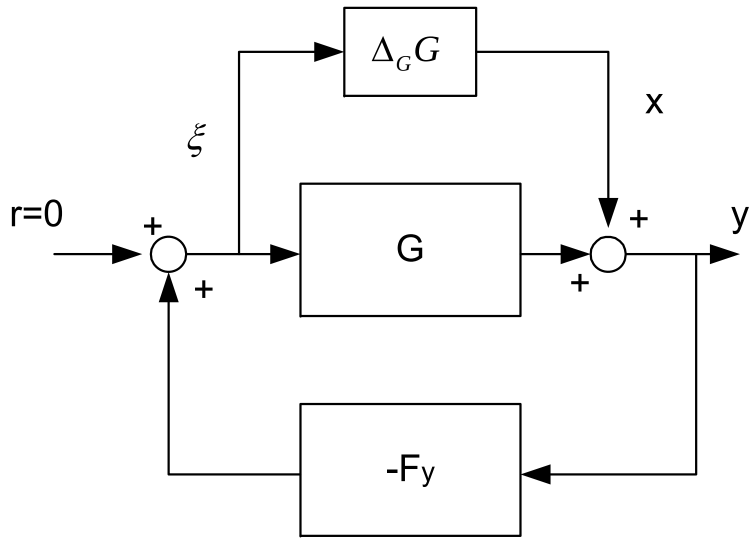
The following results can be derived

$$z_0 = (I + \Delta_z)z$$
$$\Delta_z = S_0 \Delta_G$$
$$S_0 = (I + G_0 F_y)^{-1}$$

It is seen that the **sensitivity function** S_0 shows, how the model error maps into the output error.

For those frequencies where the sensitivity function is ”small”, the effect of the modeling error in output is also small.

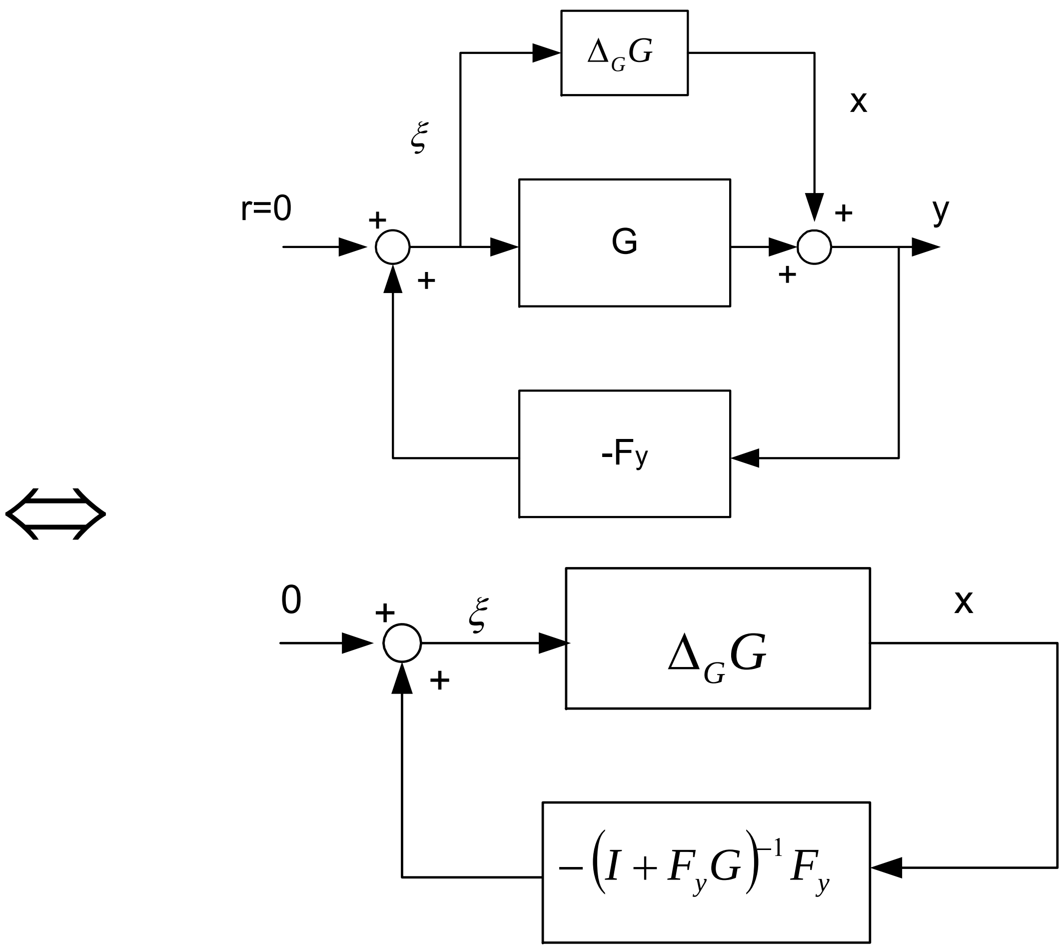
But what about stability:



Consider the loop gain, set the reference to zero, for simplicity.

Idea: study the transfer function from x to ξ

$$\xi = -F_y x - F_y G\xi \quad ; \quad \xi = -(I + F_y G)^{-1} F_y x$$



The ”Small gain theorem” guarantees that the closed loop is stable, if subsystems are stable and the gain of their product

$$\Delta_G G(I + F_y G)^{-1} F_y$$

is smaller than one. (When applying the Small gain theorem, note that the system is linear.)

Use the ”push-through”-rule

$$G(I + F_y G)^{-1} F_y = GF_y (I + GF_y)^{-1}$$
$$= (I + GF_y)^{-1} GF_y = T$$

Because Δ_G and *T* are both stable transfer functions, the system is stable if (sufficient condition)

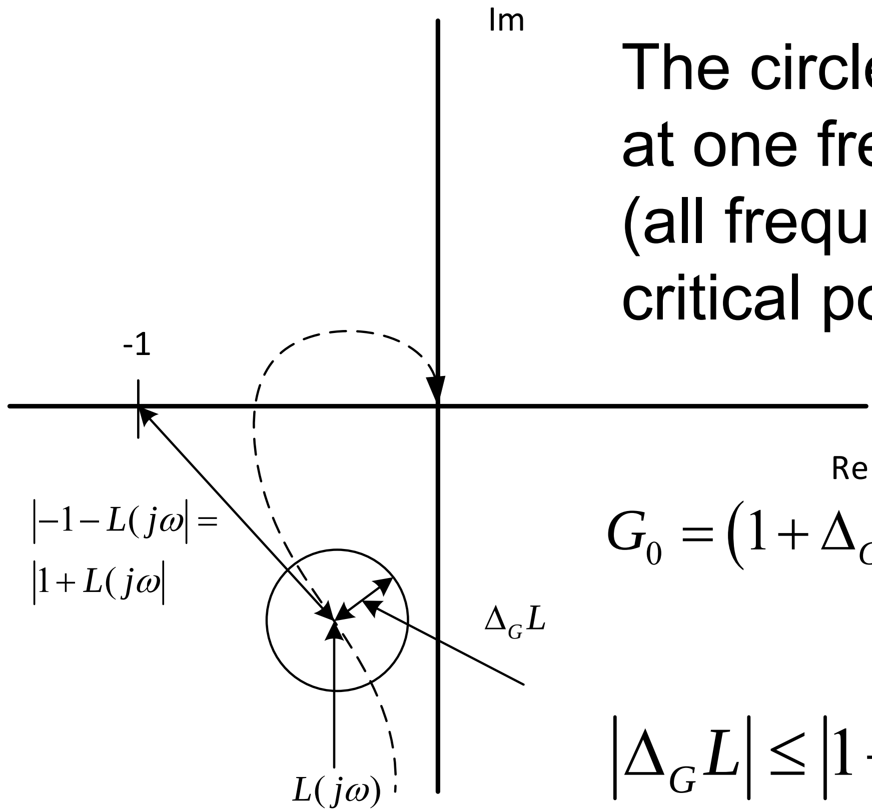
$$\left\|\Delta_G T\right\|_\infty < 1$$

which in the SISO case implies

$$\left|T(i\omega)\right| < \frac{1}{\left|\Delta_G(i\omega)\right|}, \quad \forall \omega$$

Sufficient condition for robust stability

A graphical approach (SISO)



The circle describes the uncertainty at one frequency point. The circles (all frequencies) must not cross the critical point (-1,0).

$$G_0 = (1+\Delta_G)G \Rightarrow L_0 = G_0 F_y = (1+\Delta_G)L$$

$$|\Delta_G L| \le |1+L| \Rightarrow |T| = \left|\frac{L}{1+L}\right| \le \frac{1}{|\Delta_G|}$$

Same result!

Design specifications for the closed loop system

”Design the compensator such that the controlled variable follows the reference as close as possible in spite of disturbances, measurement errors and model uncertainties. Use the control signal as little as possible.”

Let $w_u = 0$ It then holds that

$$e = (I - G_c)r - Sw + Tn$$

$$u = G_{ru} r + G_{wu}(w + n)$$

$$\Delta_z = S_0 \Delta_G$$

$$\left\| \Delta_G T \right\|_\infty < 1$$

Now it is easy to list demands for control:

1. $|I - G_c|$ ”small” , closed loop tf. close to *I*.
2. Sensitivity function *S* small, so that disturbances and model errors would have a minor impact on the output.
3. Complementary sensitivity function *T* should be small, so that measurement disturbances would not affect much and the closed loop stability would not be in danger.
4. The tfs. G_{ru} and G_{wu} should not be large.

But: $S + T = I \qquad G_c = GG_{ru}$

always hold, there are inevitable conflicts (fundamental limitations in control performance)

Static error corresponding to step input

$$e_0 = \lim_{t\to\infty} \mathrm{e(t)} = \mathrm{I - G_c(0)} = S(0) \qquad \text{(if } T = Gc\text{)}$$

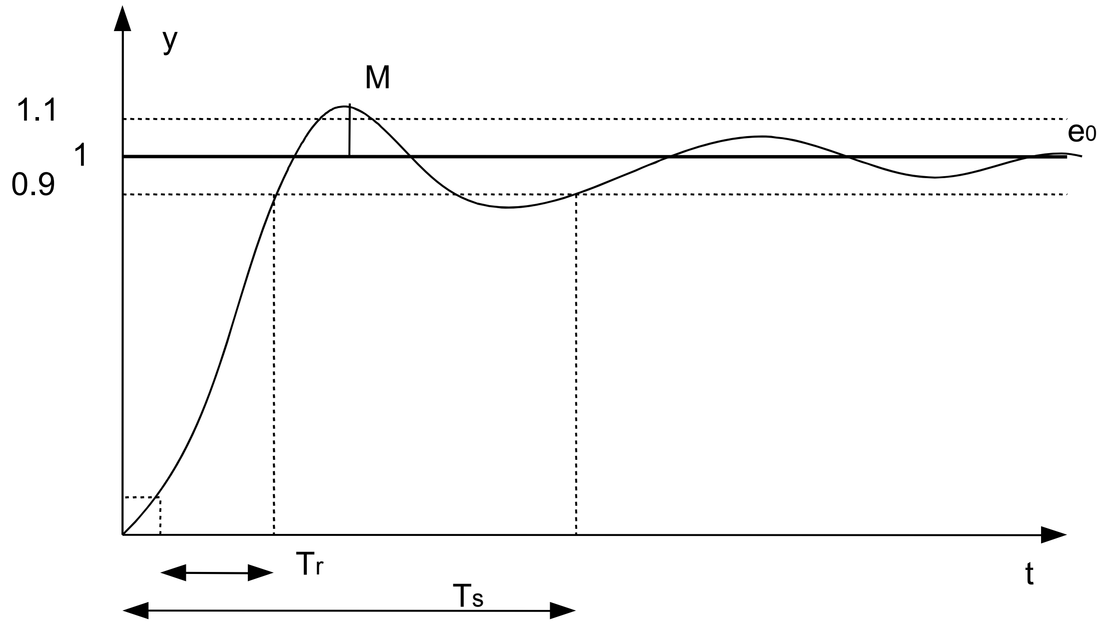
$$S_0 = (I + G_0 F_y)^{-1}$$

To minimize the static error the sensitivity must be small at low frequencies.

That means that the compensator must have high gain (e.g. integration) at low frequencies.

Other criteria: design the compensator such that Gc and S are as desired; or their poles are at desired locations.

Specifications in time domain:



M overshoot

e_0 static error

Tr rise time

Ts settling time

However, target specifications are difficult to reach by direct design methods.

Aalto University
School of Electrical Engineering

One solution: use optimal control techniques. They can in many cases guarantee immediate stability.

$$\text{Minimize} \quad \|e\|_{Q_1}^2 + \|u\|_{Q_2}^2$$

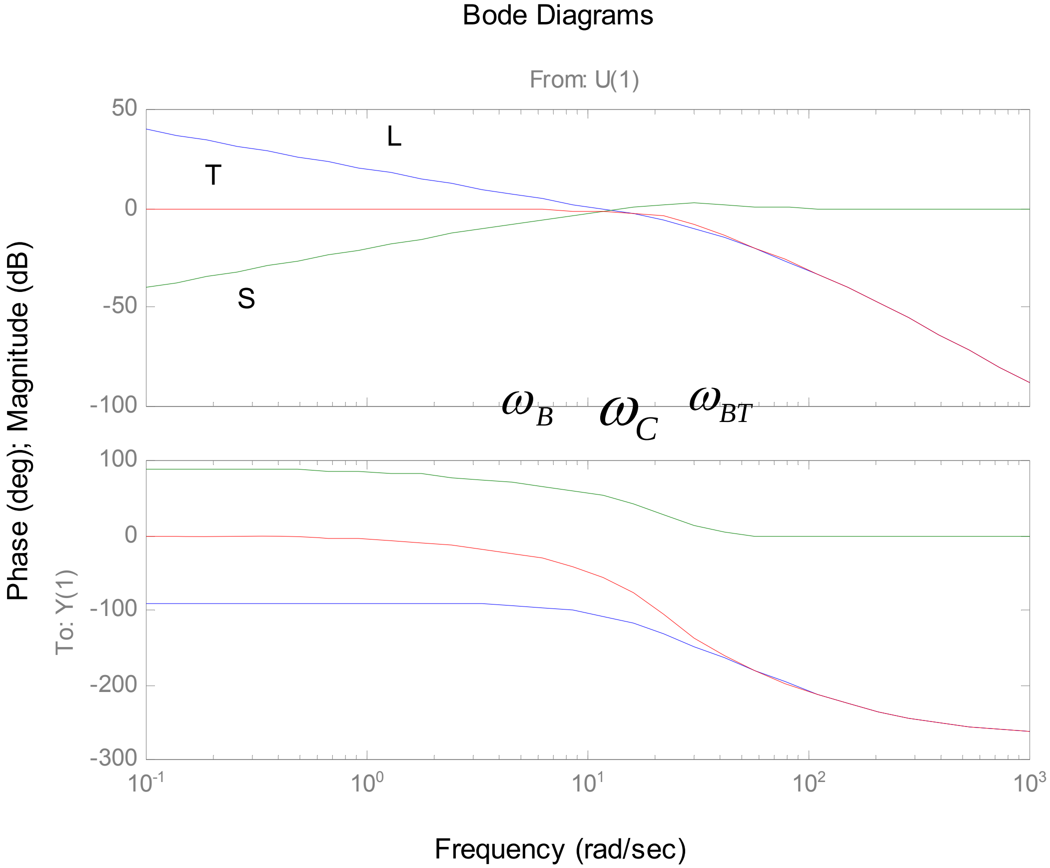
$$\text{where} \quad \|z\|_P^2 = \int_0^\infty z^T P z \, dt$$

a natural way to formalize the goal as an optimal control problem, in which the control law must be found such that the given criterion is minimized.

But: it is not straightforward to calculate a control law, that directly fulfils some time domain criteria.

Specifications in frequency domain are easier to deal with in some sense.

Frequency domain specifications:



Gain crossover ω_C

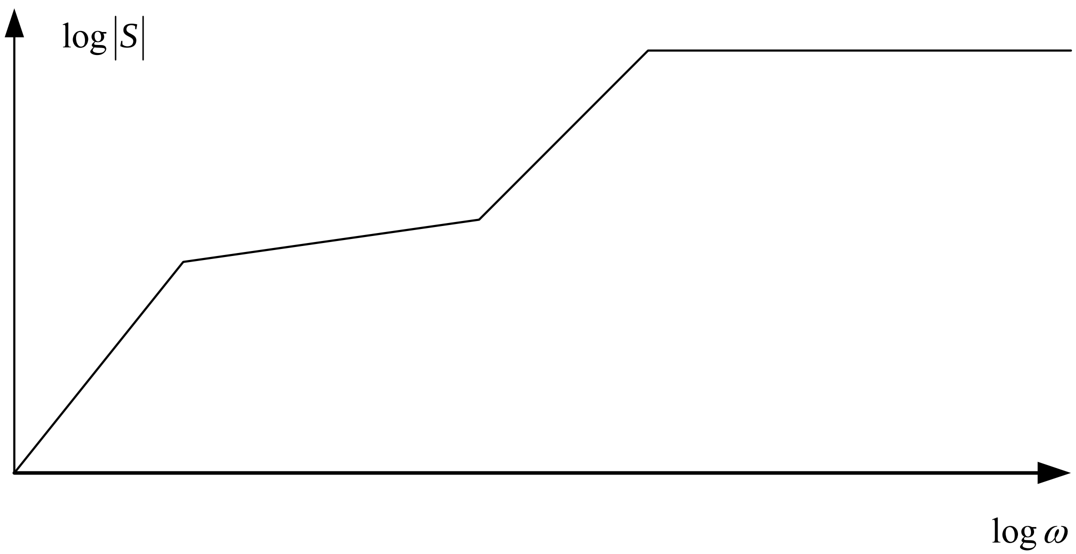
S -3 dB from ω_B below

T -3 dB from ω_{BT} above

There are different definitions for **bandwidth** (meaning the frequency range where system can follow the sinusoidal input).

Aalto University
School of Electrical Engineering

A typical S-curve



$$S + T = I$$

$|S(i\omega)| < 1$ suppress the disturbances

$$|S(i\omega)| < |W_S^{-1}(i\omega)|, \quad \forall\omega \quad \|W_S S\|_\infty \le 1$$

T small to compensate measurement disturbances, but also to guarantee robust stability.

$$|T(i\omega)| < \frac{1}{|\Delta_G(i\omega)|}, \quad \forall \omega \qquad \|W_T T\|_\infty \leq 1$$

When we want to limit the use of control signal, we set

$$\|W_u G_{ru}\|_\infty < 1$$

Design specifications: Choose the weights

$$W_S \quad W_T \quad W_u$$

and design the controller such that

$$\|W_S S\|_\infty < 1$$

$$\|W_T T\|_\infty < 1$$

$$\|W_u G_{ru}\|_\infty < 1$$

Unfortunately this is not always possible.

Instead: minimize

$$\|N\|_\infty = \underbrace{\max}_{\omega} \bar{\sigma}\big(N(j\omega)\big) < 1; \quad N = \begin{bmatrix} W_S S \\ W_T T \\ W_u G_{ru} \end{bmatrix}$$

SISO: *N* is a vector; hence

$$\bar{\sigma}(N) = \sqrt{|W_S S|^2 + |W_T T|^2 + |W_{ru} G_{ru}|^2}$$

H_∞ control problem: $\underbrace{\min}_{K} \|N(K)\|_\infty$ *K* is the compensator

Aalto University
School of Electrical
Engineering

In Matlab, the command *mixsyn* turns out to be helpful here.

```
[K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3)

 mixsyn H-infinity mixed-sensitivity synthesis method for robust
 control design. Controller K stabilizes plant G and minimizes
 the H-infinity cost function

               ||   W1*S ||
               || W2*K*S ||
               ||   W3*T ||
 where
     S := inv(I+G*K)        % sensitivity
     T := I-S = G*K/(I+G*K)  % complementary sensitivity
     W1, W2 and  W3 are stable LTI 'weights'

 Inputs:
   G         LTI plant
   W1,W2,W3  LTI weights (either SISO or compatibly dimensioned MIMO)
           To omit weight, use empty matrix (e.g., W2=[] omits W2)
```

Outputs:

- K H-infinity Controller
- CL CL=[W1*S; W2*K*S; W3*T]; weighted closed-loop system
- GAM GAM=hinfnorm(CL), closed-loop H-infinity norm
- INFO Information STRUCT, see HINFSYN documentation for details

Example:

```
G=ss(-1,2,3,4);   % plant to be controlled
  w0=10;     % desired closed-loop bandwidth
  A=1/1000;  % desired disturbance attenuation inside bandwidth
  M=2 ;      % desired bound on hinfnorm(S) & hinfnorm(T)
  s=tf('s'); % Laplace transform variable 's'
  W1=(s/M+w0)/(s+w0*A); % Sensitivity weight
  W2=[];                % Empty control weight
  W3=(s+w0/M)/(A*s+w0); % Complementary sensitivity weight
  [K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3);
```

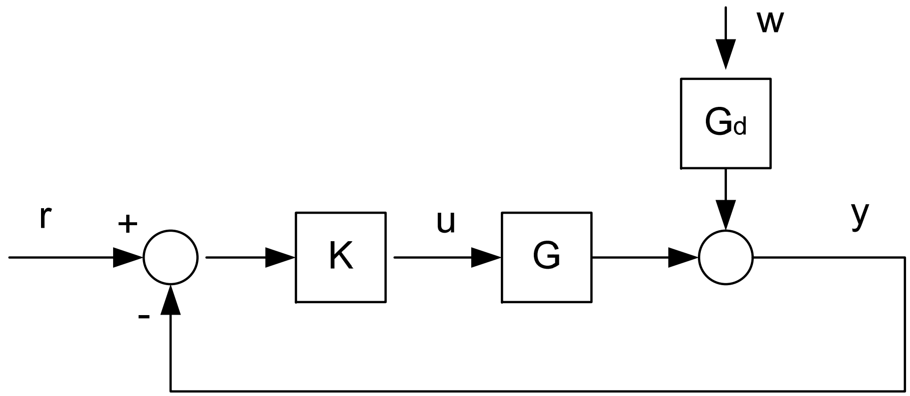
Plot results of successful design:

```
  L=G*K;  % loop transfer function
  S=inv(1+L); % Sensitivity
  T=1-S;      % complementary sensitivity
```

Mixsyn does the H infinity problem formulation automatically and solves the problem. If you use the command *hinfsyn*, you have to form the augmented plant yourself and pose the problem accordingly.

This is *Mixed Sensitivity Design*, an advanced form of *Loop Shaping Control*.

Example of control design



$$G(s) = \frac{200}{(10s+1)(0.05s+1)^2}$$

$$G_d(s) = \frac{100}{10s+1}$$

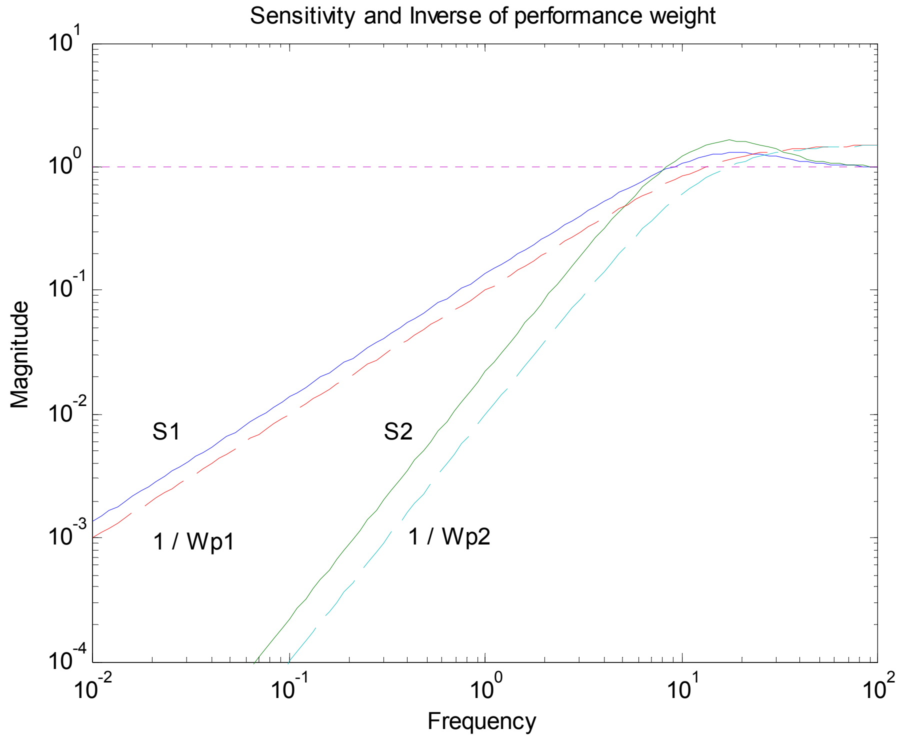
Command tracking + disturbance rejection problem

Both demands are difficult to meet simultaneously (trade-off in control design)

Let us try loop shaping by H_∞ control.

Example of control design...

```
% Mixed sensitivity design
%
% Uses the Robust Control Toolbox
%
s=tf('s');
G=200/(10*s+1)/(0.05*s+1)^2;
Gd=100/(10*s+1);
M=1.5; wb=10; A=1e-4;
Ws=tf([1/M wb], [1 wb*A]); Wu=1;
[Fy,CL,gopt]=mixsyn(G,Ws,Wu,[]);
```

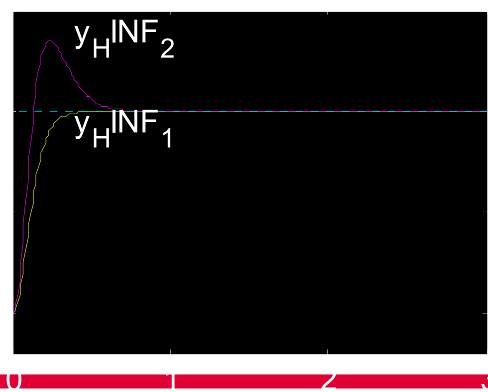


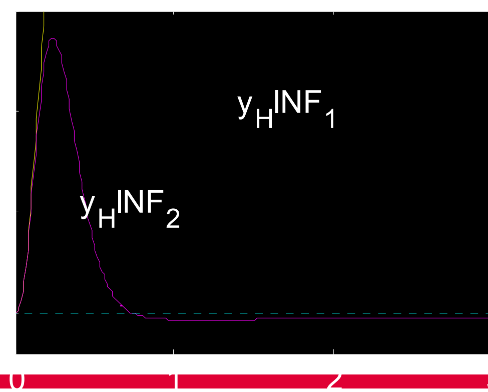
Note: $W_P = W_S$

$$W_{P1} = \frac{s/M + \omega_B^*}{s + \omega_B^* A}; \quad M = 1.5,\ \omega_B^* = 10,\ A = 10^{-4}$$

$$W_{P2} = \frac{(s/M^{1/2} + \omega_B^*)^2}{(s + \omega_B^* A^{1/2})^2}; \quad M = 1.5,\ \omega_B^* = 10,\ A = 10^{-4}$$

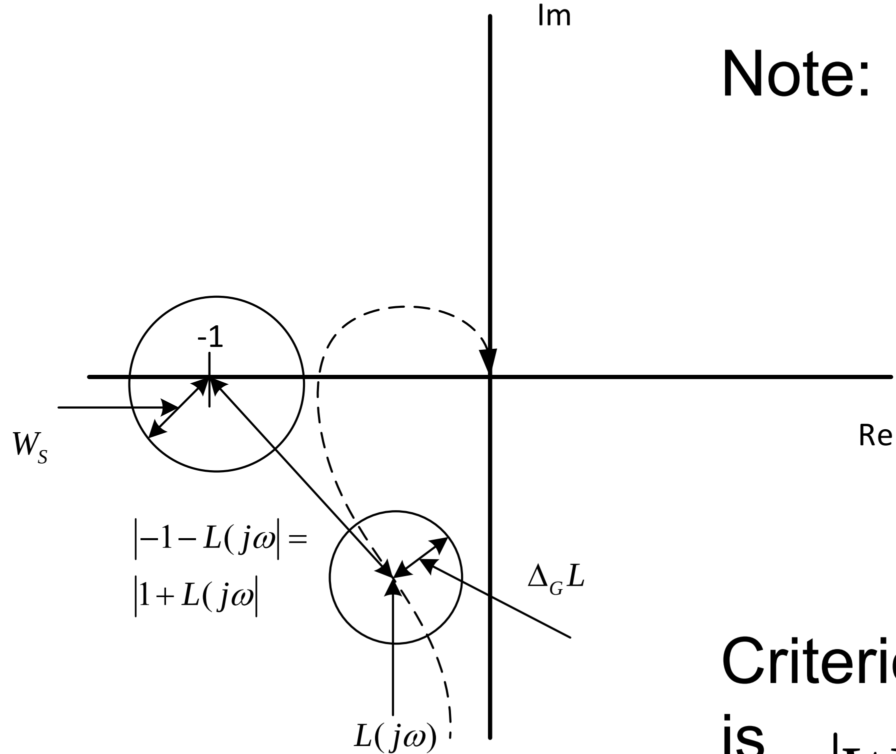
Because the load response is very poor in design 1, higher gains for the controller at low frequencies are needed (integral action).





To that end, use W_{P2} ,and the result is clearly better.

Robust performance (SISO)



Note: $|S| = \left|\dfrac{1}{1+L}\right| = \left|\dfrac{1}{1+x+jy}\right| = \dfrac{1}{\sqrt{(1+x)^2+y^2}}$

$$\Rightarrow (1+x)^2 + y^2 = \frac{1}{|S|^2}$$

The circle centered at (-1,0) represents constant values of $|S|$. The radius is $1/|S|$

Criterion for nominal performance is $|W_S S| < 1 \Rightarrow |W_S| < |1+L|$

Criterion for robust performance: all possible points in L_0 must stay outside the disk centered at (-1,0) and with the radius W_S .

$$|W_S| + |\Delta_G L| < |1+L| \Rightarrow \left|\frac{W_S}{1+L}\right| + \left|\Delta_G \frac{L}{1+L}\right| < 1 \quad \forall \omega$$

$$\max_{\omega} \left\{ |W_S S| + |\Delta_G T| \right\} < 1$$

Note: Taking $W_T = \Delta_G$ takes the condition for RP close to

$$\left\| \begin{matrix} W_S S \\ W_T T \end{matrix} \right\|_\infty = \max_{\omega} \sqrt{|W_S S|^2 + |W_T T|^2} < 1$$

So mixed sensitivity design can be used to design controllers, which have (in practice) RP.

Aalto University
School of Electrical Engineering

Main topics

Closed loop equations

$$z = G_c r + Sw - Tn + GS_u w_u$$

$$u = S_u F_r r - S_u F_y (w + n) + S_u w_u$$

Effect of model errors

$$G_0 = (I + \Delta_G)G, \qquad z = G_c r$$

$$z_0 = (I + S_0 \Delta_G) z, \qquad S_0 = (I + G_0 F_y)^{-1}$$

Robust stability , if $\quad \left\| \Delta_G T \right\|_\infty < 1$

Robust performance, if $\quad \max_\omega \left\{ |W_S S| + |\Delta_G T| \right\} < 1$