First Intermediate Exam

- First intermediate exam (IE1) on Thursday 19.10.2023, **14:00-16:00**, Classroom exam, room T3 (Computer building)
- 3 problems, max 5+5+5=15 points. Examination is done by "pen and paper". No extra material is allowed. Laplace tables are given, if they are needed. Calculators are allowed, but it is forbidden to use any advanced properties in them (e.g. matrix calculus, Laplace transformations, connection to the net etc.)
- You do not have to register to the exam.
- Note that during the examination week there are no lectures and no exercises of the course. Only the exam.

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Fundamental Limitations in Control Design

Is there a limit to how good compensator it is **possible** to design for a given process?

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Signal scaling:

Example. Room temperature dynamics

$$\dot{z}^f = K_1(x_1^f - z^f) - K_2(z^f - w^f)$$
$$\dot{x}_1^f = K_3(u^f - x_1^f) - K_4(x_1^f - z^f)$$

z is the room temperature
x_1 is the temperature of the heating radiator
w is the outdoor temperature (disturbance)
u is the temperature of the heating water (control)

The superscript f indicates that the variable is in physical (unscaled) units.

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Constants $K_1 = K_2 = K_4 = 0.7, \quad K_3 = 35$

Possible stationary point

$$x_1^f = 50^\circ C, \quad z^f = 20^\circ C, \quad w^f = -10^\circ C, \quad u^f = 50.6^\circ C$$

Purpose of control: keep room temp within $\pm 1^\circ C$ when the outdoor temp varies as $\pm 10^\circ C$;control range $\pm 20^\circ C$

In what follows the variables denote variations from the steady state.

$$z^f = \frac{0.5}{(0.03s+1)(0.7s+1)}u^f + \frac{0.01s+0.5}{(0.03s+1)(0.7s+1)}w^f$$

The time constants of the radiator and room are 0.03 and 0.7 (hours).

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Because the outdoor temperature cannot change arbitrarily fast, let us model it as

$$w^f = \frac{1}{s+1} d^f$$

where d^f is within the range $\pm 10^\circ C$

Use the scalings $u = u^f / 20, \quad z = z^f, \quad d = d^f / 10$

to obtain $z = \frac{10}{(0.03s+1)(0.7s+1)} u + \frac{0.1s+5}{(0.03s+1)(0.7s+1)(s+1)} d$

Formalize the procedure:

Physical system

$$z^f(t) = G^f(p)u^f(t) + G_d^f(p)d^f(t)$$
$$y^f(t) = z^f(t) + n^f(t)$$
$$e^f(t) = r^f(t) - z^f(t)$$

scaling matrices $D_u u = u^f, \quad D_d d = d^f$

$$Dz = z^f, \quad Dy = y^f, \quad Dn = n^f, \quad Dr = r^f, \quad De = e^f$$

”D”:s are diagonal matrices, with which different components of the variables are changed into the same scale.

Scaled system variables

$$z(t) = G(p)u(t) + G_d(p)d(t)$$
$$y(t) = z(t) + n(t)$$
$$e(t) = r(t) - z(t)$$

where $G = D^{-1}G^f D_u, \quad G_d = D^{-1}G_d^f D_d$

After proper scaling the transfer functions G and G_d are fully comparable as functions of frequency.

Earlier that would have been impossible, because the functions are related to different physical variables.

Performance limitations:

- unstable systems
- systems with delay
- non-minimum phase systems
- limitations in control signal range
- system inverse

Meaning of the system inverse

Let the system be

$$z(t) = G(p)u(t) + w(t)$$
$$y(t) = z(t) + n(t)$$

(for simplicity, assume $n = 0$)

controller $u = F_r r - F_y y$

It follows that

$$u = \frac{F_r}{1+F_y G} r - \frac{F_y}{1+F_y G} w = G^{-1}[G_c r - (1-S)w]$$

Perfect control, if $G_c = 1$ and $S = 0$

in which case $\boxed{u = G^{-1}(r - w)}$

Note. If w were measurable, this result could have been obtained directly from the system model.

$$u = G^{-1}(r - w)$$

Generally:

- perfect control means using the process inverse
- in practice, control methods are based on the search for the (approximative) inverse model
- this explains,why systems with delay and nonminimum phase systems are difficult to control

Ex. Consider the system

$$y = Gu + G_d d$$

in which the variables have been scaled such that

$$|d(t)| \le 1, \quad |u(t)| \le 1$$

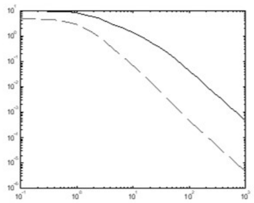
Perfect control $u = -G^{-1} G_d d$

A necessary (but not sufficient) condition for the existence of a control that compensates all allowed disturbances is

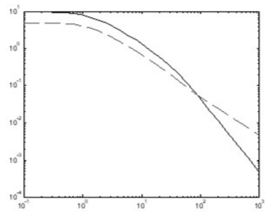
$$|G(i\omega)| \ge |G_d(i\omega)|, \quad \forall \omega$$

Let us return to the room temperature control example

$$z = \frac{10}{(0.03s+1)(0.7s+1)}u + \frac{0.1s+5}{(0.03s+1)(0.7s+1)(s+1)}d$$



s+1 with compensation Ok for all frequencies



s+1 not with compensation not perfect in high frequencies

Loop gain:

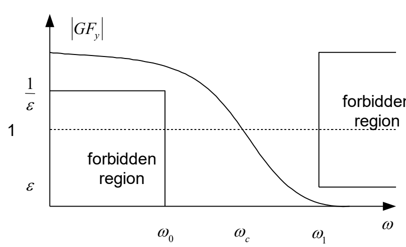
$S + T = 1$ (consider the SISO-case)

-keep S small in low frequencies
-keep T small in high frequencies

But the loop gain GF_y determines uniquely these functions

$$|S| < \varepsilon \Leftrightarrow |GF_y| > \frac{1}{\varepsilon}$$
$$|T| < \varepsilon \Leftrightarrow |GF_y| < \varepsilon$$

(approximative)



The change should be fast (as S must grow, let it happen fast in a small frequency range, in order to force T to be small).

But the loop gain and phase are interconnected!

Bode equations

For a minimum phase system it holds

$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d}{d \log \omega} \cdot \log |G(i\omega)|$$

Stability requirement: if at the gain crossover frequency the gain decreases

$20 \cdot \alpha \; dB$ (per decade), the phase is (about) $-\alpha \cdot \frac{\pi}{2}$

In order to have a positive phase margin, the gain must not drop faster than 40 dB /decade

But this is against the above requirements!

Assume that the loop gain $|L| = |GF_y|$ decreases as fast as $|s|^{-2}$ as $|s|$ tends to infinity. Then the Bode integral holds :

$$\int_0^\infty \log|S(i\omega)|\,d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\,(p_i)$$

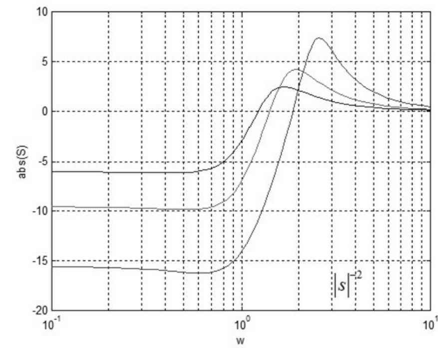
where p_i :s are the RHP poles of the loop gain $G(s)F_y(s)$ (here log means the natural logarithm (ln)).

If there are no RHP-poles, it follows

$$\int_0^\infty \log|S(i\omega)|\,d\omega = 0$$

These are again fundamental limitations.

Sometimes the phrase ”waterbed formula” is used in the literature



$$G(s) = \frac{1}{s^2 + s + 1}, \quad F_y = K$$

Note that the condition: gain $|L| = |GF_y|$ must decrease as fast as $|s|^{-2}$ as $|s|$ tends to infinity holds for practically all physical systems. (Both elements in L are at least 1st order systems).

Concequences:

1. Let the process have an unstable *pole* $p_1 > 0$

For the bandwidth the (approximative) bound

$$\omega_c > p_1$$

can be set, in order to be able to control the unstable mode.

2. Let the process have *delay* T_d

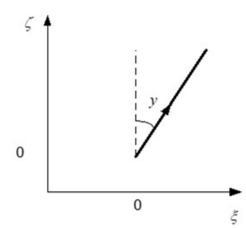
For the gain crossover frequency ω_c

$$\omega_c < (\text{appr})\, \frac{1}{T_d}$$

3. Let the process have a nonminimum phase *zero, z*

$$\omega_c < \frac{z}{2}$$

Ex. 1. Control of the inverted pendulum



control

$u = \ddot{\xi}$

$\xi + \frac{l}{2}\sin y$

$\frac{l}{2}\cos y$

Dynamic equations

$$F\cos y - mg = m\frac{d^2}{dt^2}\left(\frac{l}{2}\cos y\right) = m\frac{l}{2}(-\ddot{y}\sin y - \dot{y}^2\cos y)$$

$$F\sin y = m\ddot{\xi} + m\frac{d^2}{dt^2}\left(\frac{l}{2}\sin y\right) = mu + m\frac{l}{2}(\ddot{y}\cos y - \dot{y}^2\sin y)$$

By eliminating F $\quad \frac{l}{2}\ddot{y} - g\sin y = -u\cos y$

and linearizing with respect to small deviations

$$G(s) = \frac{-2/l}{s^2 - \frac{2g}{l}}$$

The poles are $\pm\sqrt{\frac{2g}{l}}$ (unstable)

The bandwidth should exceed $\sqrt{2g/l}$

say, $2\pi\sqrt{2g/l}$

It is seen that a short pendulum is more difficult to control than a long one.

Ex. 2. Process with delay

$$G(s) = G_1(s)e^{-sT_d}$$

Again, for the control it can be written

$$u = G^{-1}\left[G_c r - (1-S)w\right]$$

Perfect control $G_c \to 1, \quad S \to 0$ is impossible, because

it would mean

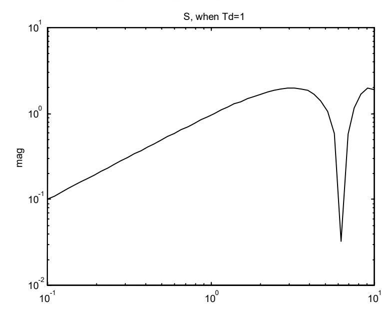
$$u = G_1^{-1}(s)e^{sT_d}(r-w)$$

which contains anticipation.

But choose ideally $G_c = e^{-sT_d}$; $T = e^{-sT_d}$ (= 1-S)
so that the delay term is cancelled from control equation.

An ”ideal” sensitivity function is then

$$S = 1 - e^{-sT_d}$$



For small frequencies

$$S(i\omega) \approx i\omega T_d$$

Then the amplitude of the (ideal) sensitivity function is smaller than one in frequencies

$$\omega < 1/T_d$$

This approximates the bandwidth, so that

$$\omega_B < \frac{1}{T_d}$$

Ex. Consider again the delay but now by means of the Padé-approximation

$$e^{-sT} \approx \frac{1 - sT/2}{1 + sT/2}$$ 1. degree Padé-approximation

But this transfer function has a non-minimum phase zero

$$z = 2/T$$

But by earlier results

$$\omega_c < \frac{z}{2} = \frac{1}{T}$$ same result!

Interpolation constraints

Let z be a RHP zero of the loop transfer function $L(z) = 0$.
Then (SISO case)

$$S(z) = \frac{1}{1 + L(z)} = 1$$ (Interpolation condition 1)

In control: $\|W_S S\|_\infty \le 1 \Leftrightarrow |S| \le \frac{1}{|W_S|}, \quad \forall\omega$

$$\Rightarrow |W_S(z)| \le 1$$

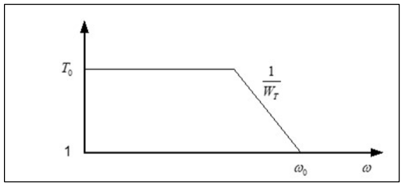
Let p_1 be a RHP pole of the loop transfer function L, $L(p_1) = \infty$
Then

$$T(p_1) = \frac{L(p_1)}{1 + L(p_1)} = \frac{1}{1 + \frac{1}{L(p_1)}} = 1$$ (Interpolation condition 2)

In control:

$$\|W_T T\|_\infty \le 1 \Leftrightarrow |T| \le \frac{1}{|W_T|}, \quad \forall\omega$$

$$\Rightarrow |W_T(p_1)| \le 1$$



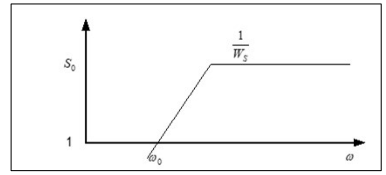
We want T to lie below this curve.

$$\|W_T T\|_\infty \le 1 \Leftrightarrow |T(i\omega)| \le \frac{1}{|W_T(i\omega)|}$$

Let $W_T = \frac{s}{\omega_0} + \frac{1}{T_0} \Rightarrow \frac{1}{W_T} = \frac{1}{(\frac{1}{T_0} + \frac{s}{\omega_0})}$ Use interpolation condition 2:

$$|W_T(p_1)| \le 1 \Rightarrow \frac{p_1}{\omega_0} + \frac{1}{T_0} \le 1 \Rightarrow \omega_0 \ge \frac{p_1}{1 - 1/T_0}$$

If we choose $T_0 = 2 \Rightarrow \omega_0 \ge 2p_1$



We want S to lie below this curve.

$$\|W_S S\|_\infty \le 1 \Leftrightarrow |S(i\omega)| \le \frac{1}{|W_S(i\omega)|}$$

Let $W_S = \frac{s + \omega_0 S_0}{S_0 s} \Rightarrow \frac{1}{W_S} = \frac{S_0 s}{s + \omega_0 S_0}$ Use interpolation condition 1:

$$|W_S(z)| \le 1 \Rightarrow \frac{z + \omega_0 S_0}{S_0 z} \le 1 \Rightarrow \omega_0 \le (1 - \frac{1}{S_0})z$$

If we accept $S_0 = 2 \Rightarrow \omega_0 \le \frac{z}{2}$

But $S_0 \to \infty \Rightarrow \omega_0 \le z$