The Perron-Frobenius theorem

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Historical remarks

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- I will state it for graphs (nlog).

The theorem

Theorem

Let G be a strongly connected graph with adjaceny matrix A. Suppose that the largest (in modulus) eigenvalue of A has modulus ρ. Then:

1. $\rho > 0$ *is an eigenvalue of A called the* Perron-Frobenius *eigenvalue;*
2. ρ *is simple;*
3. *there is an eigenvector $v > 0$, i.e., all its entries are real positive numbers, such that $Av = \rho v$;*
4. *the previous property does not hold for any other eigenvalue of A other than ρ;*
5. *if, in addition, there is an integer k such that all the entries of A^k are positive, then all eigenvalues other than ρ are strictly less than ρ.*

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- 0 as algebraic multiplicity 2 (statement 2 fails)

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Then:

- The corresponding graph is not strongly connected! (Assumption is false)
- The largest eigenvalue is 0 which is not a positive number (statement 1 fails)
- 0 as algebraic multiplicity 2 (statement 2 fails)
- The eigenvectors associated to 0 are the nonzero multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ — no way to pick a positive one (statement 3 fails)

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A has eigenvalues 1 and -1, so item 5 fails. (Exercise: check that items 1 through 4 are true though!)