2.1 Eigenvalues of the adjacency matrix

A milestone result that is useful to study eigenvalues of adjacency matrix is the Perron-Frobienus theorem, which states that a largest (in modulus) eigenvalue of a nonnegative matrix is always a nonnegative real number. While proving it is beyond the scope of the course, we will state it for the case of adjacency matrices.

We first need an additional definition; recall that a *subgraph* of a graph $G = (V, E)$ is a graph $G_s = (V_s, E_s)$ such that $V_s \subseteq V$, $E_s \subseteq E$, and the endvertices of E_s belong to V_s.

Definition 2.6. *A directed graph G is said to be* strongly connected *if, given any two vertices $i, j \in V(G)$, there exists a path on G starting from i and ending in j. An undirected graph with the same property is said to be* connected.

A strongly connected component (connected component, respectively) of a directed (undirected) graph is a maximal strongly connected (connected)subgraph.

Note that an isolated node (the empty graph with one vertex) is not connected.

Theorem 2.4. *Let G be strongly connected. Let A be the adjacency matrix of G. Suppose that the largest (in modulus) eigenvalues of A have modulus ρ. Then:*

1. *$\rho > 0$ and it is called the Perron-Frobenius eigenvalue;*
2. *ρ is simple;*
3. *there is an eigenvector v whose entries are all real and positive such that $Av = \rho v$ (and similarly for a left eigenvector w);*
4. *the previous property does not hold for eigenvector associated with other eigenvalues;*
5. *if there exists a natural number k such that A^k is positive then all the eigenvalues other than ρ are* strictly *less than ρ in modulus.*

If we remove the strongly connectedness assumption, Theorem 2.4 fails. For example, consider the graph whose adjacency matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix};$$

then its largest (and only) eigenvalue is 0 (not a positive number) and the corresponding eigenvectors are multiples of $\begin{bmatrix}1 & 0\end{bmatrix}^T$ (that cannot be made a positive vector). However, this graph is *not* strongly connected, as there is no way to go from vertex 2 to vertex 1. Still, for general graphs, a somewhat weaker results can anyway be stated.

Theorem 2.5. *Let A be the adjacency matrix of a graph G, and let the largest (in modulus) eigenvalue of A be ρ. Then:*

1. *$\rho \geq 0$ and it is called the Perron eigenvalue;*

2. *there is an eigenvector v whose entries are all real and nonnegative such that $Av = \rho v$ (and similarly for a left eigenvector w).*

Theorem 2.4 and 2.5 are, in fact, corollaries of the more general Perron-Frobenius theorem, that applies to general nonnegative matrices (not necessarily seen as adjacency matrices of a graph). In the case of simple graphs, more specific results can be given.

Proposition 2.6. *Let A be the adjacency matrix of a simple graph $G = (V, E)$, and suppose A has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let, moreover, $d_{\max}$ be the largest degree of the vertices in $V(G)$. Then,*

1. *$\lambda_1 \leq d_{\max}$;*

2. *if G is the union of the disjoint connected graphs $G_1, \ldots, G_k$, then the spectrum of A is the union of the specta of the adjacency matrices of $G_1, \ldots, G_k$.*

Proof.

1. Let v be an eigenvector associated with λ_1, so that $Av = \lambda_1 v$. By Theorem 2.5, v can be taken to be nonnegative. Moreover, being an eigenvector, is not 0. Let v_i be its maximal component; then, $v_i > 0$. Moreover,
$$\lambda_1 v_i = \sum_{j:ij\in E} v_j \leq d_{\max} \max_{j:ij\in E} v_j \leq d_{\max} v_i$$
which implies $\lambda_1 \leq d_{\max}$.

2. For $i = 1, \ldots, k$, let $G_i = (V_i, E_i)$. Up to a graph isomorphism (i.e. a relabelling of the vertices), we can assume that the vertices are ordered in such a way that $1, \ldots, \#V_1 \in V_1$, $\#V_1 + 1, \ldots, \#V_1 + \#V_2 \in V_2$, $\ldots$,

$\#V_1 + \#V_2 + \dots \#V_{k-1} + 1, \dots, \#V_1 + \#V_2 + \dots \#V_k \in V_k$. Then, by the assumptions, we have that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, which implies the result.

□

Item 2 in Proposition 2.6 can be easily generalized to the case of a graph which is the disjoint union of some connected components and some isolated points (the latter correspond to 1×1 zero matrices, which have a unique eigenvalue equal to 0). Indeed, up to a graph automorphism, A is still the direct sum of the adjacency matrices associated with either a connected component or an isolated point. Since every simple graph is the disjoint connected components and isolated points (a proof is left as an exercise), this means that for simple graph the spectral analysis of adjacency matrices can always be reduced to the case of either a connected graph or an isolated point.

For certain types of graphs, the whole spectrum can be computed explicitly.

Proposition 2.7. *Let G be a complete graph with n vertices. Then the eigenvalues of its adjacenty matrix A are*

$$\lambda_1 = n - 1 > \lambda_2 = \cdots = \lambda_n = -1.$$

Proof. Let e be the vector whose entries are all 1. Then, $A = ee^T - I$. Hence, the eigenvalues of A are the eigenvalues of ee^T, all shifted by -1. To compute the letter, we can use the matrix determinant lemma to obtain

$$\det(\lambda I - ee^T) = \det(\lambda I)(1 - \lambda^{-1} e^T I e) = \lambda^n (1 - n\lambda^{-1}) = \lambda^{n-1}(\lambda - n).$$

Hence, ee^T has eigenvalues n and 0 (the latter repeated $n - 1$ times), and thus A has eigenvalues $n-1$ and -1 (the latter having multiplicty $n-1$). □

Proposition 2.8. *Let G be a cycle graph with n vertices. Then the eigenvalues of its adjacency matrix A are (up to reordering)*

$$2 \cos \frac{2\pi j}{n}, \qquad j = 1, \dots, n.$$

Proof. The adjacency matrix A satisfies $A_{ij} = 1 \Leftrightarrow |i - j| \equiv 1 \mod n$, and hence, it is circulant with representer polynomial equal to $p(x) = x^{n-1} + x$. It is a general result in matrix theory that the eigenvalues of a circulant matrix are, for $j = 1, \ldots, n$, $\lambda_{n+1-j} = p(\omega^j)$ where $\omega = \exp\left(\frac{2\pi}{n} i\right)$. Hence, taking into account that $\omega^{jn} = 1^j = 1$, we obtain

$$\omega^j + \omega^{-j} = 2\cos\frac{2\pi j}{n}.$$

□

We conclude this section with a result on bipartite graphs.

Proposition 2.9. *Let G be a bipartite graph. Then, λ is an eigenvalue of its adjacency matrix A if and only if $-\lambda$ is.*

Proof. Let $V(G) = V_1 \cup V_2$ be the vertex partition corresponding to the bipartite property of G. Up to a graph isomorphism (i.e. a relabelling of the vertices), we can assume that the vertices are ordered in such a way that $1, \ldots, \#V_1 \in V_1$, $\#V_1 + 1, \ldots, n \in V_2$. As a consequence, the adjacency matrix has the form

$$A = \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix}$$

for some matrix $E \in \{0,1\}^{\#V_1 \times \#V_2}$. Now let (λ, v) be an eigenpair and partition $v^T = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}$ coherently with A. Then, $Av = \lambda v$ implies $Ev_2 = \lambda v_1$ and $E^T v_1 = \lambda v_2$. Therefore,

$$A \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = \begin{bmatrix} -Ev_2 \\ E^T v_1 \end{bmatrix} = -\lambda \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix},$$

proving the statement. □

2.2 Eigenvalues of the graph Laplacian

Our first result is a consequence of Theorem 2.3.

Corollary 2.10. *The graph Laplacian of a simple graph G is positive semidefinite, and it is always singular. Moreover, the multiplicity of the eigenvalue 0 is equal to the number of connected components of G.*

Proof. The positive semidefinitess is an immediate consequence of Theorem 2.3, since for any vector x we have $x^T L x = \frac{1}{2} x^T B B^T x = \frac{1}{2}\|B^T x\|_2^2 \geq 0$. Now let $e \in \mathbb{R}^n$ be the vector of all ones and let $d = \operatorname{diag}(\Delta)$ be the vector whose entries are the vertices' degrees. Observe that $(Ae)_i = \sum_{j=1}^n A_{ij}$ is equal to the degree of node i. Thus, $Le = \Delta e - Ae = d - d = 0$, and hence, 0 is an eigenvalue of L (and e is a corresponding eigenvector).

For the second part of the statement, suppose first that G is connected. Suppose that x is a vector such that $Lx = 0$; then $0 = 2x^T Lx = x^T BB^T x$ implying $B^T x = 0$. Thus, fixing an edge j having head h and tail t and using the definition of B,

$$0 = \sum_i B_{ij} x_i = x_h - x_t.$$

Therefore, the components of x are constant between any two nodes connected by any edge; as a consequence, x must also be constant between any two nodes connected by any walk. Since G is connected, we conclude that x is constant, i.e., a multiple of e. It follows that the eigenspace associated with 0 has dimension 1; hence, 0 is simple.

Now suppose that G has precisely k connected components. Then, by Proposition 2.6, L is the direct sum of k graph Laplacians corresponding to each connected component; each of these graph Laplacians is singular and has 0 as a simple eigenvalues. Hence, the multiplicitly of 0 as an eigenvalue of L is precisely k. □

An observation related to the previous Corollary (as in fact it can be used to prove semidefiniteness) is that the action of the graph Laplacian as a quadratic form has a simple expression. Indeed,

$$x^T Lx = x^T Dx - x^T Ax = \sum_{ij,ji \in E} x_i^2 - 2x_i x_j + x_j^2 = \sum_{ij,ji \in E} (x_i - x_j)^2 \geq 0.$$

We can also give an upper bound on the eigenvalues of the graph Laplacian. The bound depends on the maximal degree of the vertices in the graph, similarly to how we did it for the eigenvalues of the adjacency matrix.

Proposition 2.11. *Let G be a simple graph. The largest eigenvalue of its graph Laplacian L satisfies*

$$\lambda_1 \leq 2 \deg_{\max}.$$

Proof. Let v be an eigenvector so that $Lv = \lambda_1 v$; without loss of generality we may suppose that its largest (in absolute value) component, say, v_i, is positive (if not, replace v by $-v$). Then,

$$\lambda_1 v_i = \deg_i v_i - \sum_{ij \in E} v_j \leq \deg_{\max} v_i + \deg_{\max} v_i = 2 \deg_{\max} v_i,$$

hence the statement. $\square$

A *regular* graph is a simple graph whose vertices have all the same degree, say, d. Clearly, the graph Laplacian of a regular graph is $dI - A$, and hence, its eigenvalues are minus those of its adjacency matrix, shifted by d. This allows us to immediately find the eigenvalues of the graph Laplacian of some special graphs.

Corollary 2.12. *The graph Laplacian of the cycle of size n has eigenvalues (up to reordering)*

$$2 - 2\cos\frac{2\pi j}{n}, \qquad j = 1, \dots, n.$$

The graph Laplacian of the complete graph of size n has eigenvalues

$$\lambda_1 = \cdots = \lambda_{n-1} = n > \lambda_n = 0.$$

We conclude this section by stating a result that (via some advacend matrix theory) gives another upper bound on the eigenvalues of A and L.

Lemma 2.13. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs such that $V_1 = V_2$ and E_2 is obtained from E_1 by removing a single reciprocal edge. Let L_1, L_2 be the graph Laplacian of G_1, G_2 respectively; then their eigenvalues satisfy*

$$\lambda_i(L_2) \le \lambda_i(L_1) \le \lambda_i(L_2) + 2 \qquad i = 1, \dots, n.$$

In particular, by removing one edge from a graph, all the eigenvalues of the graph Laplacian either decrease (by at most 2) or stay the same.

Proof. Up to graph isomorphism, it is no loss of generality to assume that the edges that were removed are $12, 21$. Hence,

$$L_1 - L_2 = E := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \oplus 0_{n-2\times n-2}.$$

We now invoke Weyl's theorem on symmetric matrices. This theorem states (among other things) that, if S, T, U are three symmetric matrices of the same size n, such that $S = T + U$, and having eigenvalues $\lambda_i(S), \lambda_i(T), \lambda_i(U)$, respectively, then it holds

$$\lambda_i(T) + \lambda_n(U) \le \lambda_i(S) \le \lambda_i(T) + \lambda_1(U).$$

Let us apply this theorem to the case where $S = L_1, T = L_2, U = E$. It is easy to see that the eigenvalues of E are 2 (with multiplicity 1) and 0 (with multiplicity $n-1$). Hence,

$$\lambda_i(L_2) \leq \lambda_i(L_1) \leq \lambda_i(L_2) + 2$$

$\square$

Theorem 2.14. *Let G be a simple connected graph. The largest eigenvalue of its adjacency matrix A is $\leq n-1$, and the largest eigenvalue of its graph Laplacian L is $\leq n$.*

Proof. For A, we ijust need to apply the (previously proved) inequality $\lambda_1(A) \leq \deg_{\max} \leq n-1$.

For L, this is a consequence of Lemma 2.13 and the fact that the largest eigenvalue of the graph Laplacian of the complete graph with n vertices is precisely n. $\square$

2.3 Exercises

1. Prove that every permutation matrix is orthogonal.

2. Let P be a permutation matrix such that the ith column of P is e_j. Prove that the function $f(i) = j$ defined on all $i \in [n]$ is a bijection.

3. Let $V(G) = [n]$. Prove that the automorphisms of G are precisely the permutations of $[n]$.

4. Consider the following undirected graphs: $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 = V_2 = [7]$,

$$E_1 = \{12, 21, 13, 31, 14, 41, 15, 51, 16, 61, 17, 71\},$$

and

$$E_2 = \{34, 43, 36, 63, 45, 54, 47, 74, 56, 65, 67, 76\}.$$

Prove that G_1, G_2 have isospectral adjacency matrix while not being isomorphic to each other.

5. Compute explicitly the adjacency matrix and the graph Laplacian of the Petersen graph (see Exercises of the previous chapter). Then, prove that the eigenvalues of the former are $\{3^1, 1^5, (-2)^4\}$ and the eigenvalues of the latter are $\{5^4, 2^5, 0^1\}$, where the notation λ^m means that the eigenvalue λ has multiplicity m. *Hints: For the eigenvalues of A, compute first $A^2 + A - 2I$ and note it is of the form vv^T where v is a particular eigenvector of v: what this imply about other eigenpairs? For the eigenvalues of L, show that each node of the Peterson graph has the same degree: what can you conclude about D?*

6. Find all the eigenvalues of the adjacency matrix of the complete bipartite graph of sizes (of each component associated with the bipartite property) n_1, n_2.

7. Find all the eigenvalues of the graph Laplacian of the complete bipartite graph of sizes (of each component associated with the bipartite property) n_1, n_2.

8. Let $G = (V, E)$ be a simple graph, and let $\omega : E \mapsto \mathbb{R}/\{0\}$ be a weight function on the edges that may assume negative values. Define the graph Laplacian matrix as

$$L(G)_{i,j} = \begin{cases} 0, & (ij) \notin E, \quad i \neq j, \\ -\omega(ij), & (ij) \in E, \quad i \neq j, \\ \sum_{k:(i,k)\in E} \omega(ik), & i = j. \end{cases}$$

and let (n_-, n_0, n_+) be the *inertia* of $L(G)$, defined as the number of its negative, null and positive eigenvalues.

Let moreover G^+ be the graph obtained by removing the edges of G with negative weight. Call c^+ the number of connected components of G^+. Similarly, G^- is obtained by removing the edges with positive weight, and c^- is the number of connected components of G^-.

Prove that

- $c^- \leq n_0 + n_+$,
- $c^- - c^+ \leq n_+$.

Hint: Use the following Interlacing Eigenvalue Theorem.
For every real symmetric matrix $M = M^T \in \mathbb{R}^{n\times n}$ and every non-zero vector $v \in \mathbb{R}^n$,

$$\lambda_i(M) \leq \lambda_i(M + vv^T), \quad \forall i = 1, \dots, n,$$

$$\lambda_{i+1}(M) \geq \lambda_i(M + vv^T), \quad \forall i = 1, \dots, n-1.$$