Lecture 9: The conditional choice probability (CCP) method: Finite Dependence and Examples

Ciprian Domnisoru
Aalto University

Back to the dynamic discrete choice problem

- $\max \text{E}\left(\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^t d_{jt}[u_j(s_t|\theta)+\varepsilon_{jt}]|s_1\right)$

- expectations are taken with respect to the joint distribution of future states and the ε_{jt}
- Future states are not affected by ε except through current and past choices $E(s_{t+1}|d_t,...d1,\varepsilon_t,...,\varepsilon_1)=E(s_{t+1}|d_t,...,d_1)$
- Let $d^0_{jt}(s_t,\varepsilon_t)$ represent the optimal decision rule in period t conditional on s_t and ε_t
- Define $\bar{V}(s_t)$, the integrated value function, as the expected payoff associated with being in state s_t, assuming optimal choices from period t onward.

$$\bar{V}(s_t)=E[\sum_{\tau=t+1}^{T}\sum_{j=1}^{J}\beta^{\tau-t}d^0_{j\tau}(s_\tau,\varepsilon_\tau)[u_j(s_\tau|\theta)+\varepsilon_{j\tau}|s_t]]$$

Emax and conditional value functions

$$\bar{V}(s_t) = E[\sum_{\tau=t+1}^{T}\sum_{j=1}^{J}\beta^{\tau-t} d_{j\tau}^{0}(s_\tau,\varepsilon_\tau)[u_j(s_\tau|\theta)+\varepsilon_{j\tau}|s_t]]$$

- The discounted sum of expected payoffs just before ε_t is revealed conditional on behaving according to the optimal decision rule.
- According to Bellman's principle of optimality, we can write the Emax (expectation of the solution of an optimization problem) expression, which involves two multiple-dimensional integrals.

$$\bar{V}(s_t) = E[\sum_{j=1}^{J} d_{j\tau}^{0}\left(u_j(s_\tau|\theta)+\varepsilon_{j\tau}+\beta\int V_{t+1}(s_{t+1})f(s_{t+1})|s_t,d_{j\tau}^{0})\right)|s_t]$$

$$\bar{V}(s_t) = \sum_{j=1}^{J}\int d_{j\tau}^{0}\left(u_j(s_\tau|\theta)+\varepsilon_{j\tau}+\beta\int V_{t+1}(s_{t+1})f(s_{t+1})|s_t,d_{j\tau}^{0})\right)dF_\varepsilon(\varepsilon_\tau)$$

- Define the choice specific conditional value functions:

$$v_k(s_t) = u_k(s_t)+\beta E[\bar{V}(s_{t+1}|s_t)]$$

Choice specific value functions

$$
\begin{aligned}
v_k(s_t) &\equiv u_k(s_t) + \beta E[\bar{V}(s_{t+1})|s_t] \\
&= u_k(s_t) + E\Big(\sum_{\tau=t+1}^{T} \sum_{j=1}^{J} \beta^{\tau-t} d^0_{j\tau}(s_\tau, \varepsilon_\tau)[u_j(s_\tau|\theta) + \varepsilon_{j\tau}] \; |s_t] \Big) \\
&= u_k(s_t) + \sum_{\tau=t+1}^{T} \sum_{j=1}^{J} \beta^{\tau-t} p_j(s_\tau)[u_j(s_\tau|\theta) + E[\varepsilon_{j\tau}|d_{j\tau} = 1, s_\tau]]
\end{aligned}
$$

- $p_j(s_\tau)$ represents the **conditional choice probability** for choice j in period τ when the state is s_τ
- Express the multidimensional integral over the error terms in terms of the choice probabilities
- $p_{jt}(s_t) = E[d^0_{jt}(s_t, \varepsilon)|s_t] = \int d^0_{jt}(s_t, \varepsilon) g_t(\varepsilon|s_t) d\varepsilon$

The Hotz-Miller conditional choice probability representation

- Assume we have J choices, with $u_j = v_j + \varepsilon_j$, with v_j a set of functions whose form is known (up to a vector of unknown parameters) and $\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\varepsilon_J \sim F(\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\varepsilon_J)$. Choice 1 is selected when $v_1 + \varepsilon_1 > v_k + \varepsilon_k$, or $\varepsilon_k < v_1 - v_k + \varepsilon_1$ for all k=2...J

$$\begin{cases} p_1 = & \int_{-\infty}^{\infty} \int_{-\infty}^{v_1 - v_2 + \varepsilon_1} .... \int_{-\infty}^{v_1 - v_J + \varepsilon_1} f(\varepsilon_1, \varepsilon_2, \varepsilon_3, ...\varepsilon_J) d\varepsilon_2 .....d\varepsilon_J d\varepsilon_1, \\ & = \phi_1(v_1 - v_2, v_1 - v_3, ..., v_1 - v_J; F) \end{cases}$$

- We can write similar expressions for any p_j and create a system of J-1 equations.
- Hotz and Miller (1993) apply the inverse function theorem to this system and obtain J-1 solution functions $v_1 - v_k = \psi_{1k}(p_2, ...p_J)$
- Once you have J-1 solution functions for any base choice (e.g., the first), you can easily translate to another: $v_j - v_k = \psi_{jk}(p)$, where $p = (p_1, ...p_J)$
- This shows that in general the choice probabilities can be mapped into differences in the conditional valuations, relative to an arbitrary base.

The inversion mapping for the basic MNL

- Consider a RUM with payoffs $u_j = v_j + \varepsilon_j$
- $p_j = \frac{exp(v_j)}{\sum_k exp(v_k)}$
- Implies $log p_j - log p_k = v_j - v_k$
- Thus, for the MNL, we can specialize $v_j - v_k = \psi_{jk}(p)$ to $\psi_{jk}(p) = log p_j - log p_k = log(p_j / p_k)$

Choice specific value functions

- Consider the "selection bias"/correction term $E[\varepsilon_{j\tau}|d_{j\tau}=1,s_\tau]$. Note that $v_k(s_t)+\varepsilon_{kt}$ is the value of choosing alternative k in period t when the state is s_t
- Write the "selection bias" term as a functions of the choice probabilities $E[\varepsilon_{kt}|d_{kt},s_t] = w_k(\psi(p(s_t)))$
- This allows us to rewrite:

$$v_k(s_t) = u_k(s_t) + E\left(\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{\tau-t}p_j(s_\tau)[u_j(s_\tau|\theta) + w_j(\psi(p(s_\tau))|s_t = s, d_{kt} = 1\right)$$

- If we can estimate the functions $p(s_\tau)$, and we know the $w_j(\psi(p(s_\tau))$ functions(as is true under Type I extreme value errors for MNL and GEV models) then we can express $v_k(s_t)$ in terms of current and future flow utility functions and choice probabilities.

The inversion mapping for the basic MNL

- Arcidiacono and Miller (2007) show that the $w_j(\psi(\hat{p}(s_\tau))$ function can be simplified if errors have a Type I extreme value distribution:
- For the basic multinomial logit: E$[\varepsilon_j \mid d_j = 1]$= $\gamma - logp_j$,
- where γ=0.577 is Euler's constant
- In our problem, $\sum_j p_j(v_j + E(\varepsilon_j|d_j = 1])$=

  $\sum_j p_j(v_j + \gamma - logp_j) = \gamma + \sum_j p_j(v_j - log\left(\frac{expv_j}{\sum_k expv_k}\right)) = \gamma +$

  $\sum_j p_j log(\sum_k expv_k) = \gamma + log(\sum_k expv_k)$

Example: Dynamic structural model of occupational choice

Beginning at age 16, individuals choose among a set of mutually exclusive and exhaustive options, with the goal of maximizing lifetime utility:

$$\max_{c_a} E[\sum_{a=16}^{65} \beta^{a-16}(u[s_a, c_a] + \varepsilon[c_a])]$$

Their decision updates the state space (s_a includes includes schooling, work experience, previous period choices), and the process is repeated at age 17 and thereafter. The choices are:

- continuing K-12 education
- enrollment in 2-year college
- enrollment in 4-year college in a STEM, Healthcare or Education major
- enrollment in 4-year college in a Liberal Arts, Social Sciences or Business major
- enrollment in 4-year college, undeclared major
- enrollment in graduate school
- employment in the college sector
- employment in the non-college sector
- home production

Choice-specific utility functions

Choice- specific utility functions:

$$U_{ca} = \alpha_c + \gamma_c X_c + \varepsilon_{ca},$$

X_c- alternative-specific vector of covariates.

- Employment choices: **expected log compensation** (net of predicted student loan repayments), experience, years in college in different majors, degrees attained, college selectivity
- Education choices: **expected log tuition** .
- All choices: previous period choices, unemployment rate, cumulative GPA, AFQT percentile, Black, Hispanic

ε_{ca}- idiosyncratic shocks, distributed Type I extreme value
α_c - choice specific constant (preference, endowment)- allowed to differ among two types in model with heterogeneity.

Bellman representation

$$\max_{c_a} E[\sum_{a=16}^{65} \beta^{a-16}(u[s_a, c_a] + \varepsilon[c_a])]$$ can be represented as:

$$V_a(s_a, \varepsilon(c_a)) = \max_{c_a}[u_a(s_a, c_a) + \varepsilon(c_a) \; + \beta \int V_{a+1} f(s_{a+1} \mid s_a, c_a) dG(\varepsilon(c_{a+1}))]$$

Hotz and Miller (1993) representation: The conditional value function:

$$v_a(s_a, c_a) = u_a(s_a, c_a) + \beta \int V_{a+1}(s_{a+1}, \varepsilon(c_{a+1})) f(s_{a+1} \mid s_a, c_a) dG(\varepsilon(c_{a+1}))$$

can be replaced by:

$$v_a(s_a, c_a) = u_a(s_a, c_a) + \beta[ln[\sum_{j=1}^{J} exp(v_{a+1}(s_{a+1}, c_{a+1} = j))] + \gamma] f(s_{a+1} \mid s_a, c_a)$$

Proof

Goal: find a closed form solution for $\int \sum_{s_{a+1}} V_{a+1}(s_{a+1}, \varepsilon(c_{a+1})) f(s_{a+1} \mid s_a, c_a) dG(\varepsilon(c_{a+1}))$

$$V_a(s_a, \varepsilon(c_a)) = \max_{c_a}[v_a(s_a, c_a) + \varepsilon(c_a)]$$

If $Y_1, Y_2 \ldots\ Y_C$ are independent, non-identically distributed extreme value random variables with location parameters α_1, $\alpha_2, \ldots, \alpha_C$ and common scale parameter σ, the distribution of Y_c is given by:

$$F(x \mid \alpha_c, \sigma) = P(Y_c \leq x \mid \alpha_c, \sigma) = exp[-exp(\frac{-(x-\alpha_c)}{\sigma})]$$

$\varepsilon(c_a)$ is a Type I extreme value random variable. $v_a(s_a, c_a) + \varepsilon(c_a)$ is also an extreme value random variable, with location parameter v_a .

Proof

$$P(\max_c Y_c \leq x) = \Pi_{c=1}^{C} P(Y_c \leq x) = \Pi_{c=1}^{C} exp[-exp[\frac{-(x-\alpha_c)}{\sigma}]] =$$

$$= exp\{\sum_c -exp\{\frac{-(x-\alpha_c)}{\sigma}\}\} = exp\{-exp(\frac{-x}{\sigma})\sum_c exp(\frac{\alpha_c}{\sigma})\}$$

$$= exp[-exp[\frac{-(x-\sigma ln \sum exp\frac{\alpha_c}{\sigma})}{\sigma}]]$$

Mean of the Type I distribution $E(\varepsilon) = \alpha + \sigma\gamma$, location parameter $\alpha = ln[\sum_{c=1}^{C} exp(v_c/\sigma)]$

Hence $\int \sum_{s_{a+1}} V_{a+1}(s_{a+1}, \varepsilon(c_{a+1})) f(s_{a+1} \mid s_a, c_a) dG(\varepsilon(c_{a+1})) =$

$$= \sum_{s_{a+1}} [ln[\sum_{j=1}^{J} exp(v_{a+1}(s_{a+1}, c_{a+1} = j)] + \gamma] f(s_{a+1} \mid s_a, c_a)$$

Model representation-CCP

Given the Type I extreme value assumption, the probability of choosing any option $\tilde{c}_a$ at age a is:

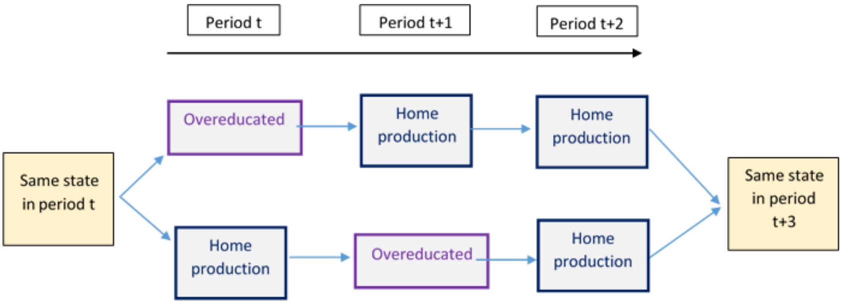
$$Pr(c_a = \tilde{c}_a \mid s_a) = \frac{exp(v_a(s_a, c_a = \tilde{c}))}{\sum_{j=1}^{J} exp(v_a(s_a, c_a = j))} = \frac{1}{\sum_{j=1}^{J} exp(v_a(s_a, c_a = j) - (v_a(s_a, c_a = \tilde{c}))}.$$

The value function can be expressed as:

$$v_a(s_a, c_a = \tilde{c}) = u_a(s_a, c_a = \tilde{c}) +$$
$$\beta[ln[\sum_{j=1}^{J} exp((v_{a+1}(s_{a+1}, c_{a+1} = j) - (v_{a+1}(s_{a+1}, c_{a+1} = q))]f(s_{a+1} \mid s_a, c_a = \tilde{c}) +$$
$$\beta[v_{a+1}(s_{a+1}, c_{a+1} = q)]f(s_{a+1} \mid s_a, c_a = \tilde{c})$$

We can use a telescoping argument, and similarly express v_{a+1} and v_{a+2} as functions of conditional choice probabilities. However, we would still need to evaluate v_{a+3}. For "terminal" and "renewal" problems the choice of base can lead us to a situation where $v_{a+1} = 0$. For other problems, we will employ the finite dependence method.

Finite dependence: Arcidiacono and Miller (2011)



- In period t+3, the conditional value functions v_{a+3} and probabilities
- $Pr^{-1}(c_{a+3})$ will be the same, and will drop out when we take the difference in value functions at period t.

Estimation

$$v_a(s_a, d_a = \tilde{c}) - v_a(s_a, d_a = H) = u_a(s_a, d_a = \tilde{c}) +$$
$$\beta \ln[Pr^{-1}(c_{a+1} = H \mid s_{a+1})] f(s_{a+1} \mid s_a, c_a = \tilde{c}) -$$
$$\beta \ln[Pr^{-1}(c_{a+1} = \tilde{c} \mid s_{a+1})] f(s_{a+1} \mid s_a, c_a = H) -$$
$$\beta [u_{a+1}(s_{a+1}, c_{a+1} = \tilde{c})] f(s_{a+1} \mid s_a, c_a = H) +$$
$$\beta^2 \ln[Pr^{-1}(c_{a+2} = H \mid s_{a+2})] f(s_{a+2} \mid s_{a+1}, c_{a+1} = H) f(s_{a+1} \mid s_a, c_a = \tilde{c}) -$$
$$\beta^2 \ln[Pr^{-1}(c_{a+2} = H \mid s_{a+2})] f(s_{a+2} \mid s_{a+1}, c_{a+1} = \tilde{c}) f(s_{a+1} \mid s_a, c_a = H)$$

1. Estimate conditional choice probabilities using a flexible multinomial logit.
2. Calculate the scalar terms along the finite dependence path.
3. Estimate the flow utility parameters using a multinomial logit with offset terms equal to the scalars calculated in 2).
4. Heterogeneity introduced through a finite mixture model (Heckman and Singer, 1984) with two types, through an adaptation of the EM algorithm (Arcidiacono and Miller, 2011).

Identification assumptions: Assumptions imposed on the idiosyncratic shocks; Normalization of the utility of the home production option; Discount factor set to 0.9

Heterogeneity analysis

- Population composed of a finite mixture of K types, whose type K probabilities are fixed.
- Sample log likelihood is then given by:
  $ln\mathscr{L}(\Theta)= \sum_{i=1}^{N} ln(\sum_{k=1}^{K} \prod_{a=1}^{A} \pi_k \mathscr{L}_i)$
- Maximizing the above logarithm would require integrating out over the unobserved states. Instead, Arcidiacono and Miller (2011) adaptation of the EM algorithm reinstates additive separability by treating the unobserved states as observed in the maximization step, and maximizing the expected log likelihood function instead:
  $ln\mathscr{L}(\Theta) = \sum_{i=1}^{N} \sum_{k=1}^{K} \sum_{a=1}^{A} q_{ik}^{j} \mathscr{L}_i(c_a \mid s_a, k, \pi^{(j)}, \hat{p}^{(j)}, \theta^{(j-1)})$

Back

Estimation: EM algorithm

Step 1. After using an initial guess, update $q_{ik}^{(j+1)}$ conditional on the data and parameter values, using Bayes' rule:

$$q_{ik}^{(j+1)} = \frac{\prod_{a=1}^{A} \pi_k^{(j)} \mathscr{L}_{ika}(c_a|s_a, k, \pi^{(j)}, \hat{p}^{(j)}, \theta^{(j)})}{\sum_{k=1}^{K} \prod_{a=1}^{A} \pi_k^{(j)} \mathscr{L}_{ika}(c_a|s_a, k, \pi^{(j)}, \hat{p}^{(j)}, \theta^{(j)})}$$

Step 2. Given $q_{ik}^{(j+1)}$, update $\pi_{ik}^{(j+1)} = \frac{1}{N} \sum_{i=1}^{N} q_{ik}^{(j+1)}$.

Step 3. Maximize the expected likelihood function to obtain new estimates $\theta^{(j+1)}$, given $q_{ik}^{(j+1)}$, $\pi_{ik}^{(j+1)}$, $\hat{p}^{(j)}$, and c_a and s_a. At the maximization step, $q_{ik}^{(j+1)}$ are treated as given, acting as population weights.

Step 4. Update the conditional choice probability parameters to $p^{(j+1)}$ using the conditional likelihood of observing choices $\tilde{c}$ when the parameters are θ^{j+1} , $\hat{p}^{j}$:
$p_{\tilde{c}}^{(j+1)} = P(c = \tilde{c} \mid s, k, \theta^{(j+1)}, \pi^{j})) = \mathscr{L}_{\tilde{c}}(s, k, \theta^{(j+1)}, \pi^{j})$

Kang, Lowery and Wardlaw (2014)

- Dynamic model of the decision to close a troubled bank: immediate costs versus delayed (contagion) risk.
- FDIC as a utility-maximizing agent who faces a variety of costs when closing an insolvent bank. In each period, an FDIC regulator chooses whether to close a bank. If the bank is closed, the regulator must pay a monetary cost. This cost is the direct payout from the insurance fund required to make depositors whole.
- "Among other things, we find some evidence of political influence on the FDIC, and we find that the FDIC tended to prefer to allow the largest and smallest banks to continue to operate even at an expected cost to taxpayers."

Kang, Lowery and Wardlaw paper

We estimate the model using the Hotz and Miller (1993) technique, which permits identification of structural parameters of dynamic models without solving the dynamic programming problem as is required in methods following Rust (1987). Instead of solving for the potentially highly intractable value function, the researcher acknowledges that the agent has solved the problem in order to choose his optimal behavior, and, therefore, there is an invertible mapping between value functions and empirically observed choice probabilities. This approach leads to an estimator with a much lower computational burden compared to traditional methods, allowing us to consider a richer set of explanatory variables and several different variations of the

Kang, Lowery and Wardlaw paper

This assumption allows identification of the monetary and nonmonetary costs of an orderly bank closure even though we never observe disorderly failures. This identification is perhaps surprising because the fear of a disorderly closure is the primary reason that banks are closed in an orderly fashion at all. Our partial identification from limited data demonstrates a powerful aspect of the estimation technique applied here. As long as we can obtain estimates of the expected closure cost one period ahead, the probability of closure given each state, and the state transition process, we can identify nonmonetary cost parameters and the discount factor without needing either data or assumptions about the costs along choice paths farther into the future. This feature arises because we apply the inversion theorem from Hotz and Miller (1993) to represent value-function differences as ratios of logs of choice probabilities, allowing us to account for the dynamics in the decision process of the FDIC without actually calculating the value function directly.

Kang, Lowery and Wardlaw paper

The insight that permits identification of the dynamic model is that it is possible to substitute out the terms that cannot be expressed in closed form by replacing them with estimated closure probabilities that can be estimated from the data. This is possible because these closure probabilities are the outcome of the agent solving the dynamic programming problem. As a result, we can obtain an expression that should hold (Equation 6), and that contains only observable data, preestimated quantities, and structural parameters.

Kang, Lowery and Wardlaw paper: terminal actions

Note that once $d_{it} = 1$ is chosen, this action cannot be undone and there are no more actions to follow in future periods. Hence, our model features a "terminal action" or "absorbing state" from the dynamic discrete choice literature. We see this by observing that $V_1(x_{it})$ does not include any future payoffs.