# 13 Heat engines and the second law



In this chapter, we introduce the second law of thermodynamics, probably the most important and far-reaching of all concepts in thermal physics. We are going to illustrate it with an application to the theory of 'heat engines', which are machines that produce work from a temperature difference between two reservoirs.[1] It was by considering such engines that nineteenth century physicists such as Carnot, Clausius and Kelvin came to develop their different statements of the second law of thermodynamics. However, as we will see in subsequent chapters, the second law of thermodynamics has a wider applicability, affecting all types of processes in large systems and bringing insights in information theory and cosmology. In this chapter, we will begin by stating two alternative forms of the second law of thermodynamics and then discuss how these statements impact on the efficiency of heat engines.

[1]A **reservoir** in this context is a body which is sufficiently large that we can consider it to have essentially infinite heat capacity. This means that you can keep sucking heat out of it, or dumping heat into it, without its temperature changing. See Section 4.6.

## 13.1 The second law of thermodynamics

The second law of thermodynamics can be formulated as a statement about the direction of heat flow that occurs as a system approaches equilibrium (and hence there is a connection with the direction of the 'arrow of time'). Heat is always observed to flow from a hot body to a cold body, and the reverse process, in isolation,[2] never occurs. Therefore, following Clausius, we can state the second law of thermodynamics as follows:

[2]The 'in isolation' phrase is very important here. In a refrigerator, heat is sucked out of cold food and squirted out of the back into your warm kitchen, so that it flows in the 'wrong' direction: from cold to hot. However, this process is not happening in isolation. Work is being done by the refrigerator motor and electrical power is being consumed, adding to your electricity bill.

> **Clausius' statement of the second law of thermodynamics**:
> 'No process is possible whose sole result is the transfer of heat from a colder to a hotter body.'

It turns out that an equivalent statement of the second law of thermodynamics can be made, concerning how easy it is to change energy between different forms, in particular between work and heat. It is very easy to convert work into heat. For example, pick up a brick of mass $m$ and carry it up to the top of a building of height $h$ (thus doing work on it equal to $mgh$) and then let it fall back to ground level by dropping it off the top (being careful not to hit passing pedestrians). All the work that you've done in carrying the brick to the top of the building will be dissipated in heat (and a small amount of sound energy) as the brick hits the ground. However, conversion of heat into work is much harder, and in fact the complete conversion of heat into work is impossible. This point is expressed in Kelvin's statement of the second law of thermodynamics:

**Kelvin's statement of the second law of thermodynamics**:
'No process is possible whose sole result is the complete conversion of heat into work.'

These two statements of the second law of thermodynamics do not seem to be obviously connected, but the equivalence of these two statements will be proved in Section 13.4.

## 13.2 The Carnot engine

Kelvin's statement of the second law of thermodynamics says that you can't *completely* convert heat into work. However, it does not forbid *some* conversion of heat into work. How good a conversion from heat to work is possible? To answer this question, we have to introduce the concept of an engine. We define an **engine** as a system operating a cyclic process that converts heat into work. It has to be cyclic so that it can be continuously operated, producing a steady power.

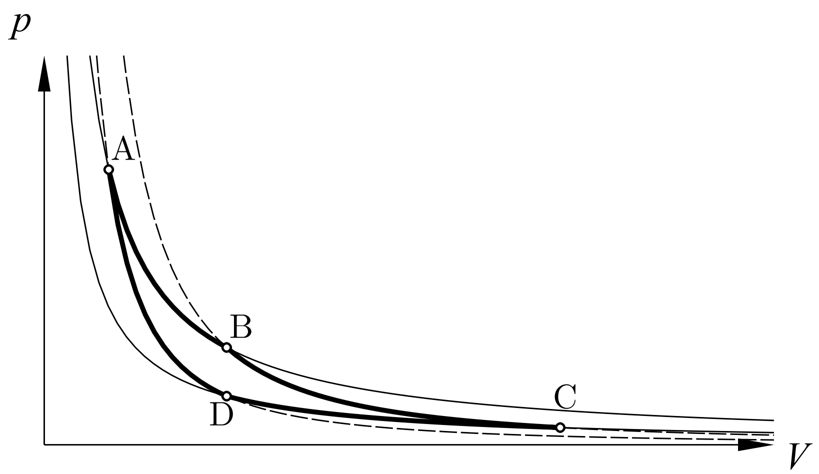


**Fig. 13.1** A Carnot cycle consists of two reversible adiabats (BC and DA) and two reversible isotherms (AB and CD). The Carnot cycle is here shown on a $p$–$V$ plot. It is operated in the direction A→B→C→D→A, i.e. clockwise around the solid curve. Heat $Q_\mathrm{h}$ enters in the isotherm A→B and heat $Q_\ell$ leaves in the isotherm C→D.

One such engine is the **Carnot engine**, which is based upon a process called a **Carnot cycle** and which is illustrated in Figure 13.1. An equivalent plot which is easier to sketch is shown in Figure 13.2. The Carnot cycle consists of two reversible adiabats and two reversible isotherms for an ideal gas. The engine operates between two heat reservoirs, one at the high temperature of $T_\mathrm{h}$ and one at the lower temperature of $T_\ell$. Heat enters and leaves only during the reversible isotherms (because no heat

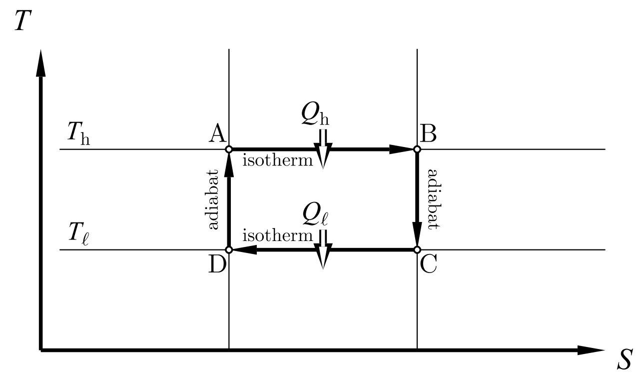


**Fig. 13.2** A Carnot cycle can be drawn on replotted axes where the isotherms are shown as horizontal lines ($T$ is constant for an isotherm) and the adiabats are shown as vertical lines (where the quantity $S$, which must be some function of $pV^\gamma$, is constant in an adiabatic expansion; in Chapter 14 we will give a physical interpretation of $S$).

can enter or leave during an adiabat). Heat $Q_h$ enters during the expansion A→B and heat $Q_\ell$ leaves during the compression C→D. Because the process is cyclic, the change of internal energy (a state function) in going round the cycle is zero. Hence the work output by the engine, $W$, is given by

$$W = Q_h - Q_\ell. \tag{13.1}$$

**Example 13.1**

Find an expression for $Q_h/Q_\ell$ for an ideal gas undergoing a Carnot cycle in terms of the temperatures $T_h$ and $T_\ell$.

*Solution*:

Using the results of Section 12.2, we can write down

$$\mathrm{A} \rightarrow \mathrm{B}: \quad Q_h = RT_h \ln \frac{V_B}{V_A}, \tag{13.2}$$

$$\mathrm{B} \rightarrow \mathrm{C}: \left(\frac{T_h}{T_\ell}\right) = \left(\frac{V_C}{V_B}\right)^{\gamma-1}, \tag{13.3}$$

$$\mathrm{C} \rightarrow \mathrm{D}: \quad Q_\ell = -RT_\ell \ln \frac{V_D}{V_C}, \tag{13.4}$$

$$\mathrm{D} \rightarrow \mathrm{A}: \left(\frac{T_\ell}{T_h}\right) = \left(\frac{V_A}{V_D}\right)^{\gamma-1}. \tag{13.5}$$

Equations 13.3 and 13.5 lead to

$$\frac{V_B}{V_A} = \frac{V_C}{V_D}, \tag{13.6}$$

and dividing eqn 13.2 by eqn 13.4 and substituting in eqn 13.6 leads to

$$\boxed{\frac{Q_h}{Q_\ell} = \frac{T_h}{T_\ell}.} \tag{13.7}$$

This is a key result.[3]

[3]In fact, when we later prove in Section 13.3 that all reversible engines have this efficiency, one can use eqn 13.7 as a thermodynamic *definition* of temperature. In this book, we have preferred to define temperature using a statistical argument via eqn 4.7.

The Carnot engine is shown schematically in Fig. 13.3. It is drawn as a machine with heat input $Q_\mathrm{h}$ from a reservoir at temperature $T_\mathrm{h}$, drawn as a horizontal line, and two outputs, one of work $W$ and the other of heat $Q_\ell$ which passes into the reservoir at temperature $T_\ell$.

The concept of **efficiency** is important to characterize engines. It is the ratio of 'what you want to achieve' to 'what you have to do to achieve it'. For an engine, what you want to achieve is work (to pull a train up a hill for example) and what you have to do to achieve it is to put heat in (by shovelling coal into the furnace), keeping the hot reservoir at $T_\mathrm{h}$ and providing heat $Q_\mathrm{h}$ for the engine. We therefore define the efficiency $\eta$ of an engine as the ratio of the work out to the heat in. Thus

$$\eta = \frac{W}{Q_\mathrm{h}}. \qquad (13.8)$$

Note that since the work out cannot be greater than the heat in (i.e. $W < Q_\mathrm{h}$) we must have that $\eta < 1$. The efficiency must be below 100%.

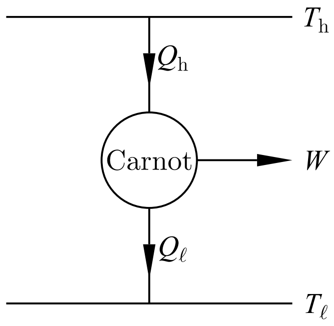


**Fig. 13.3** A Carnot engine shown schematically. In diagrams such as this one, the arrows are labelled with the heat/work flowing in one cycle of the engine.

**Example 13.2**

For the Carnot engine, the efficiency can be calculated using eqns 13.1, 13.7 and 13.8 as follows: substituting eqn 13.1 into 13.8 yields

$$\eta_\mathrm{Carnot} = \frac{Q_\mathrm{h} - Q_\ell}{Q_\mathrm{h}}, \qquad (13.9)$$

and eqn 13.7 then implies that

$$\eta_\mathrm{Carnot} = \frac{T_\mathrm{h} - T_\ell}{T_\mathrm{h}} = 1 - \frac{T_\ell}{T_\mathrm{h}}. \qquad (13.10)$$

How does this efficiency compare to that of a real engine? It turns out that real engines are much less efficient than Carnot engines.

**Example 13.3**

A power station steam turbine operates between $T_\mathrm{h} \sim 800$ K and $T_\ell =$ 300 K. If it were a Carnot engine, it could achieve an efficiency of $\eta_\mathrm{Carnot} = (T_\mathrm{h} - T_\ell)/T_\mathrm{h} = 60\%$, but in fact real power stations do not achieve the maximum efficiency and figures closer to 40% are typical.

## 13.3 Carnot's theorem

The Carnot engine is in fact the most efficient engine possible! This is stated in Carnot's theorem, as follows:

**Carnot's theorem**:
Of all the heat engines working between two given temperatures, none is more efficient than a Carnot engine.

Remarkably, one can *prove* Carnot's theorem on the basis of Clausius' statement of the second law of thermodynamics.[4] The proof follows a *reductio ad absurdum* argument.

[4]This means that Carnot's theorem is, in itself, a statement of the second law of thermodynamics.

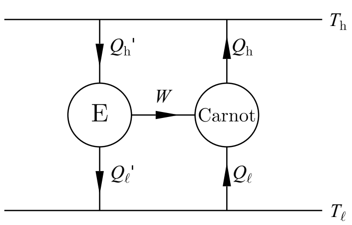


**Fig. 13.4** A hypothetical engine E, which is more efficient than a Carnot engine, is connected to a Carnot engine.

**Proof:** Imagine that E is an engine which is more efficient than a Carnot engine (i.e. $\eta_{\mathrm{E}} > \eta_{\mathrm{Carnot}}$). The Carnot engine is reversible so one can run it in reverse. Engine E, and a Carnot engine run in reverse, are connected together as shown in Fig. 13.4. Now since $\eta_{\mathrm{E}} > \eta_{\mathrm{Carnot}}$, we have that

$$\frac{W}{Q'_{\mathrm{h}}} > \frac{W}{Q_{\mathrm{h}}}, \tag{13.11}$$

and so

$$Q_{\mathrm{h}} > Q'_{\mathrm{h}}. \tag{13.12}$$

The first law of thermodynamics implies that

$$W = Q'_{\mathrm{h}} - Q'_{\ell} = Q_{\mathrm{h}} - Q_{\ell}, \tag{13.13}$$

so that

$$Q_{\mathrm{h}} - Q'_{\mathrm{h}} = Q_{\ell} - Q'_{\ell}. \tag{13.14}$$

Now $Q_{\mathrm{h}} - Q'_{\mathrm{h}}$ is positive because of eqn 13.12, and therefore so is $Q_\ell - Q'_\ell$. The expression $Q_{\mathrm{h}} - Q'_{\mathrm{h}}$ is the net amount of heat dumped into the reservoir at temperature $T_{\mathrm{h}}$. The expression $Q_\ell - Q'_\ell$ is the net amount of heat extracted from the reservoir at temperature $T_\ell$. Because both these expressions are positive, the combined system shown in Fig. 13.4 simply extracts heat from the reservoir at $T_\ell$ and dumps it into the reservoir at $T_{\mathrm{h}}$. This violates Clausius' statement of the second law of thermodynamics, and therefore engine E cannot exist.

**Corollary**:
All reversible engines have the same efficiency $\eta_{\mathrm{Carnot}}$.

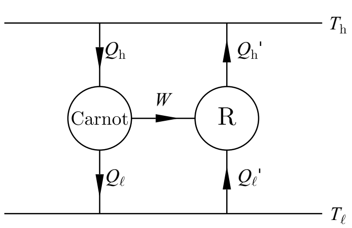


**Fig. 13.5** A hypothetical reversible engine R is connected to a Carnot engine.

**Proof:** Imagine another reversible engine R. Its efficiency $\eta_{\mathrm{R}} \leq \eta_{\mathrm{Carnot}}$ by Carnot's theorem. We run it in reverse and connect it to a Carnot engine going forwards, as shown in Figure 13.5. This arrangement will simply transfer heat from the cold reservoir to the hot reservoir and violates Clausius' statement of the second law of thermodynamics *unless* $\eta_{\mathrm{R}} = \eta_{\mathrm{Carnot}}$. Therefore all reversible engines have the same efficiency

$$\eta_{\mathrm{Carnot}} = \frac{T_{\mathrm{h}} - T_\ell}{T_{\mathrm{h}}}. \tag{13.15}$$

## 13.4 Equivalence of Clausius and Kelvin statements

We first prove the proposition that if a system violates Kelvin's statement of the second law of thermodynamics, it violates Clausius' statement of the second law of thermodynamics.

**Proof:** If a system violates Kelvin's statement of the second law of thermodynamics, one could connect it to a Carnot engine as shown in Figure 13.6. The first law implies that

$$Q'_{\mathrm{h}} = W \tag{13.16}$$

and that

$$Q_{\mathrm{h}} = W + Q_\ell. \tag{13.17}$$

The heat dumped in the reservoir at temperature $T_{\mathrm{h}}$ is

$$Q_{\mathrm{h}} - Q'_{\mathrm{h}} = Q_\ell. \tag{13.18}$$

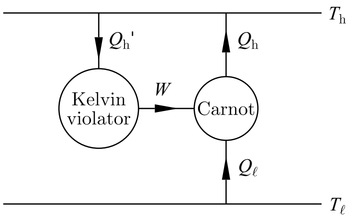


**Fig. 13.6** A Kelvin violator is connected to a Carnot engine.

This is also equal to the heat extracted from the reservoir at temperature $T_\ell$. The combined process therefore has the net result of transferring heat $Q_\ell$ from the reservoir at $T_\ell$ to the reservoir at $T_{\mathrm{h}}$ as its sole effect and thus violates Clausius' statement of the second law of thermodynamics. Therefore the Kelvin violator does not exist.

We now prove the opposite proposition, that if a system violates Clausius' statement of the second law of thermodynamics, it violates Kelvin's statement of the second law of thermodynamics.

**Proof:** If a system violates Clausius' statement of the second law of thermodynamics, one could connect it to a Carnot engine as shown in Figure 13.7. The first law implies that

$$Q_{\mathrm{h}} - Q_\ell = W. \tag{13.19}$$

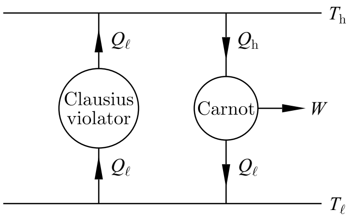


**Fig. 13.7** A Clausius violator is connected to a Carnot engine.

The sole effect of this process is thus to convert heat $Q_{\mathrm{h}} - Q_{\mathrm{l}}$ into work and thus violates Kelvin's statement.

We have thus shown the equivalence of Clausius' and Kelvin's statements of the second law of thermodynamics.

## 13.5 Examples of heat engines

One of the first engines to be constructed was made in the first century by Hero of Alexandria, and is sketched in Fig. 13.8(a). It consists of an airtight sphere with a pair of bent pipes projecting from it. Steam is fed via another pair of pipes and once expelled through the bent pipes causes rotational motion. Though Hero's engine convincingly converts heat into work, and thus qualifies as a *bona fide* heat engine, it was little more than an entertaining toy. More practical was the engine sketched in Fig. 13.8(b) which was designed by Thomas Newcomen (1664–1729).

This was one of the first practical steam engines and was used for pumping water out of mines. Steam is used to push the piston upwards. Then, cold water is injected from the tank and condenses the steam, reducing the pressure in the piston. Atmospheric pressure then pushes the piston down and raises the beam on the other side of the fulcrum. The problem with Newcomen's engine was that one had then to heat up the steam chamber again before steam could be readmitted and so it was extremely inefficient. James Watt (1736–1819) famously improved the design so that condensation took place in a separate chamber which was connected to the steam cylinder by a pipe. This work led the foundation of the industrial revolution.

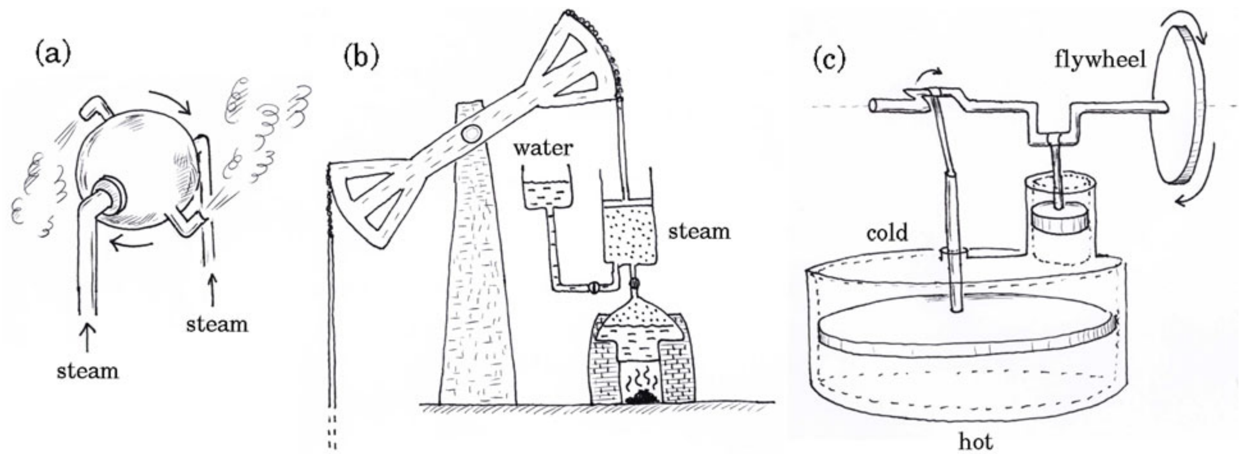


**Fig. 13.8** Sketches of (a) Hero's engine, (b) Newcomen's engine and (c) Stirling's engine.

Another design of an engine is **Stirling's engine**, the brainchild of the Rev. Robert Stirling (1790–1878) and which is sketched in Fig. 13.8(c), It works purely by the repeated heating and cooling of a sealed amount of gas. In the particular engine shown in Fig. 13.8(c), the crankshaft is driven by the two pistons in an oscillatory fashion, but the 90° bend ensures that the two pistons move out of phase. The motion is driven by a temperature differential between the top and bottom surfaces of the engine. The design is very simple and contains no valves and operates at relatively low pressures. However, such an engine literally has to 'warm up' to establish the temperature differential and so it is harder to regulate power output.

One of the most popular engines is the **internal combustion engine** used in most automobile applications. Rather than externally heating water to produce steam (as with Newcomen's and Watt's engines) or to produce a temperature differential (as with Stirling's engine), here the burning of fuel inside the engine's combustion chamber generates the high temperature and pressure necessary to produce useful work. Different fuels can be used to drive these engines, including diesel, gasoline, natural gas and even biofuels such as ethanol. These engines all produce carbon dioxide, and this has important consequences for Earth's atmosphere, as we shall discuss in Chapter 37. There are many different types of internal combustion engines, including piston engines (in which pressure is converted into rotating motion using a set of pistons), combustion turbines (in which gas flow is used to spin a turbine's blades) and jet engines (in which a fast moving jet of gas is used to generate thrust).[5]

[5] In Exercise 13.5 we consider the Otto cycle, which models the diesel engine, a type of internal combustion engine.

## 13.6 Heat engines running backwards

In this section we discuss two applications of heat engines in which the engine is run in reverse, putting in work in order to move heat around.

**Example 13.4**

(a) **The refrigerator:**
The refrigerator is a heat engine which is run backwards so that you put work in and cause a heat flow from a cold reservoir to a hot reservoir (see Figure 13.9). In this case, the cold reservoir is the food inside the refrigerator which you wish to keep cold and the hot reservoir is usually your kitchen. For a refrigerator, we must define the efficiency in a different way from the efficiency of a heat engine. This is because what you want to achieve is 'heat sucked out of the contents of the refrigerator' and what you have to do to achieve it is 'electrical work' from the mains electricity supply. Thus we define the efficiency of a refrigerator as

$$\eta = \frac{Q_\ell}{W}. \tag{13.20}$$

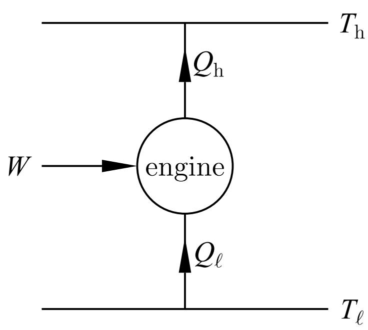


**Fig. 13.9** A refrigerator or a heat pump. Both devices are heat engines run in reverse (i.e. reversing the arrows on the cycle shown in Fig. 13.3).

For a refrigerator fitted with a Carnot engine, it is then easy to show that

$$\eta_\mathrm{Carnot} = \frac{T_\ell}{T_\mathrm{h} - T_\ell}, \tag{13.21}$$

which can yield an efficiency above 100%.

(b) **The heat pump:**
A heat pump is essentially a refrigerator (Figure 13.9 applies also for a heat pump), but it is utilized in a different way. It is used to pump heat from a reservoir, to a place where it is desired to add heat. For example, the reservoir could be the soil/rock several metres underground and heat could be pumped out of the reservoir into a house which needs heating. In one cycle of the engine, we want to add heat $Q_\mathrm{h}$ to the house, and now $W$ is the work we must apply (in the form of electrical work) to accomplish this. The efficiency of a heat pump is therefore defined as

$$\eta = \frac{Q_\mathrm{h}}{W}. \tag{13.22}$$

Note that $Q_\mathrm{h} > W$ and so $\eta > 1$. The efficiency is always above 100%! (See Exercise 13.1.) This shows why heat pumps are attractive[6] for heating. It is always possible to turn work into heat with 100% efficiency (an electric fire turns electrical work into heat in this way), but a heat pump can allow you to get even more heat into your house for the same electrical work (and hence for the same electricity bill!).

[6] However, the capital cost means that heat pumps have not become popular until recently.

For a heat pump fitted with a Carnot engine, it is easy to show that

$$\eta_\mathrm{Carnot} = \frac{T_\mathrm{h}}{T_\mathrm{h} - T_\ell}. \tag{13.23}$$

## 13.7 Clausius' theorem

Consider a Carnot cycle. In one cycle, heat $Q_\mathrm{h}$ enters and heat $Q_\ell$ leaves. Heat is therefore not a conserved quantity of the cycle. However, we found in eqn 13.7 that for a Carnot cycle

$$\frac{Q_\mathrm{h}}{Q_\ell} = \frac{T_\mathrm{h}}{T_\ell}, \tag{13.24}$$

and so if we define[7] $\Delta Q_\mathrm{rev}$ as the heat entering the system at each point, we have that

$$\sum_\mathrm{cycle} \frac{\Delta Q_\mathrm{rev}}{T} = \frac{Q_\mathrm{h}}{T_\mathrm{h}} + \frac{(-Q_\ell)}{T_\ell} = 0, \tag{13.25}$$

[7] The subscript 'rev' on $\Delta Q_\mathrm{rev}$ is there to remind us that we are dealing with a reversible engine.

and so $\Delta Q_\mathrm{rev}/T$ is a quantity which sums to zero around the cycle. Replacing the sum by an integral, we could write

$$\oint \frac{đQ_\mathrm{rev}}{T} = 0 \tag{13.26}$$

for this Carnot cycle.

Our argument so far has been in terms of a Carnot cycle which operates between two heat distinct reservoirs. Real engine cycles can be much more complicated than this in that their 'working substance' changes temperature in a much more complicated way and, moreover, real engines do not behave perfectly reversibly.[8] Therefore we would like to generalize our treatment so that it can be applied to a general cycle operating between a whole series of reservoirs and we would like the cycle to be either reversible or irreversible. Our general cycle is illustrated in Fig. 13.10(a). For this cycle, heat $đQ_i$ enters at a particular part of the cycle. At this point the system is connected to a reservoir which is at temperature $T_i$. The total work extracted from the cycle is $\Delta W$, given by

$$\Delta W = \sum_\mathrm{cycle} đQ_i, \tag{13.27}$$

[8] You need to get the energy out of a real engine quickly, so you do not have time to everything quasistatically!

from the first law of thermodynamics. The sum here is taken around the whole cycle, indicated schematically by the dotted circle in Fig. 13.10(a).

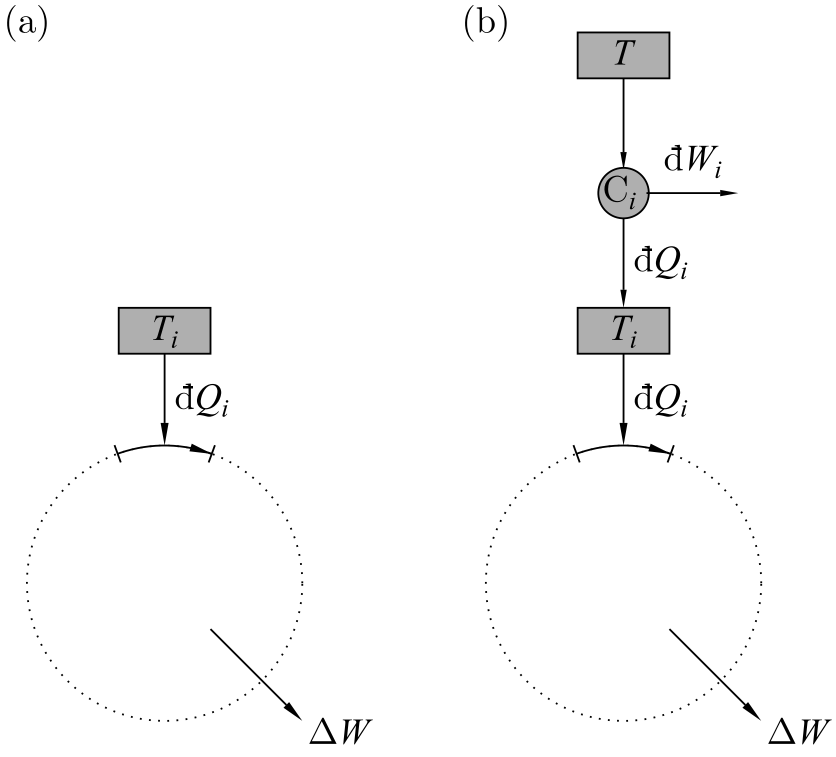


**Fig. 13.10** (a) A general cycle in which heat $đQ_i$ enters in part of the cycle from a reservoir at temperature $T_i$. Work $\Delta W$ is extracted from each cycle. (b) The same cycle, but showing the heat $đQ_i$ entering the reservoir at $T_i$ from a reservoir at temperature $T$ via a Carnot engine (labelled $C_i$).

Next we imagine that the heat at each point is supplied via a Carnot engine which is connected between a reservoir at temperature $T$ and the reservoir at temperature $T_i$ (see Fig. 13.10(b)). The reservoir at $T$ is common for all the Carnot engines connected at all points of the cycle. Each Carnot engine produces work $đW_i$, and for a Carnot engine we know that

$$\frac{\text{heat to reservoir at } T_i}{T_i} = \frac{\text{heat from reservoir at } T}{T}, \qquad (13.28)$$

and hence

$$\frac{đQ_i}{T_i} = \frac{đQ_i + đW_i}{T}. \qquad (13.29)$$

Rearranging, we have that

$$đW_i = đQ_i \left( \frac{T}{T_i} - 1 \right). \qquad (13.30)$$

The thermodynamic system in Fig. 13.10(b) looks at first sight to do nothing other than convert heat to work, which is not allowed according to Kelvin's statement of the second law of thermodynamics, and hence we must insist that this is not the case. Hence

$$\text{total work produced per cycle} = \Delta W + \sum_{\text{cycle}} đW_i \leq 0. \qquad (13.31)$$

Using eqns 13.27, 13.30 and 13.31, we therefore have that

$$T \sum_{\text{cycle}} \frac{đQ_i}{T_i} \leq 0. \qquad (13.32)$$

Since $T > 0$, we have that

$$\sum_{\text{cycle}} \frac{đQ_i}{T_i} \leq 0, \qquad (13.33)$$

and replacing the sum by an integral, we can write this as

$$\oint \frac{đQ}{T} \leq 0, \qquad (13.34)$$

which is known as the **Clausius inequality**, embodied in the expression of Clausius' theorem:

> **Clausius' theorem**:
> For any closed cycle, $\oint \frac{đQ}{T} \leq 0$, where equality necessarily holds for a reversible cycle.

## Chapter summary

- No process is possible whose sole result is the transfer of heat from a colder to a hotter body. (Clausius' statement of the second law of thermodynamics)
- No process is possible whose sole result is the complete conversion of heat into work. (Kelvin's statement of the second law of thermodynamics)
- Of all the heat engines working between two given temperatures, none is more efficient than a Carnot engine. (Carnot's theorem)
- All the above are equivalent statements of the second law of thermodynamics.
- All reversible engines operating between temperatures $T_\mathrm{h}$ and $T_\ell$ have the efficiency of a Carnot engine: $\eta_{\mathrm{Carnot}} = (T_\mathrm{h} - T_\ell)/T_\mathrm{h}$.
- For a Carnot engine:
$$\frac{Q_\mathrm{h}}{Q_\ell} = \frac{T_\mathrm{h}}{T_\ell}.$$
- Clausius' theorem states that for any closed cycle, $\oint \frac{đQ}{T} \leq 0$ where equality necessarily holds for a reversible cycle.

# Further reading

An entertaining account of how steam engines really work may be found in Semmens and Goldfinch (2000). A short account of Watt's development of his engine is Marsden (2002).

# Exercises

(13.1) A heat pump has an efficiency greater than 100%. Does this violate the laws of thermodynamics?

(13.2) What is the maximum possible efficiency of an engine operating between two thermal reservoirs, one at 100°C and the other at 0°C?

(13.3) The history of science is littered with various schemes for producing **perpetual motion**. A machine which does this is sometimes referred to as a *perpetuum mobile*, which is the Latin term for a perpetual motion machine.

- A perpetual motion machine of the first kind produces more energy than it uses.
- A perpetual motion machine of the second kind produces exactly the same amount of energy as it uses, but it continues running forever indefinitely by converting all its waste heat back into mechanical work.

Give a critique of these two types of machine and state which laws of thermodynamics they each break, if any.

(13.4) A possible ideal-gas cycle operates as follows:
(i) from an initial state $(p_1, V_1)$ the gas is cooled at constant pressure to $(p_1, V_2)$;
(ii) the gas is heated at constant volume to $(p_2, V_2)$;
(iii) the gas expands adiabatically back to $(p_1, V_1)$.
Assuming constant heat capacities, show that the thermal efficiency is

$$1 - \gamma \frac{(V_1/V_2) - 1}{(p_2/p_1) - 1}. \qquad (13.35)$$

(You may quote the fact that in an adiabatic change of an ideal gas, $pV^\gamma$ stays constant, where $\gamma = c_p/c_V$.)

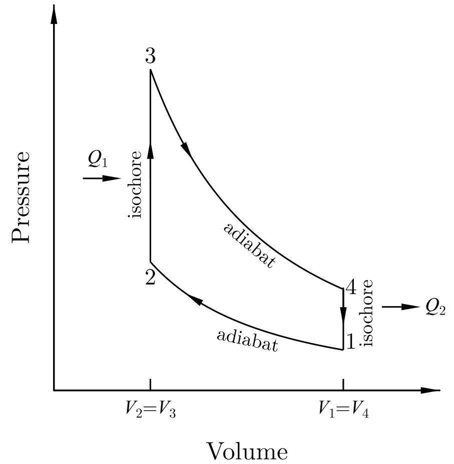


**Fig. 13.11** The Otto cycle.

(13.5) Show that the efficiency of the standard Otto cycle (shown in Fig. 13.11) is $1 - r^{1-\gamma}$, where $r = V_1/V_2$ is the compression ratio. The **Otto cycle** is the four-stroke cycle in internal combustion engines in cars, lorries and electrical generators.

(13.6) An ideal air conditioner operating on a Carnot cycle absorbs heat $Q_2$ from a house at temperature $T_2$ and discharges $Q_1$ to the outside at temperature $T_1$, consuming electrical energy $E$. Heat leakage into the house follows Newton's law,

$$Q = A[T_1 - T_2], \qquad (13.36)$$

where $A$ is a constant. Derive an expression for $T_2$ in terms of $T_1$, $E$ and $A$ for continuous operation when the steady state has been reached.
The air conditioner is controlled by a thermostat. The system is designed so that with the thermostat set at 20°C and outside temperature 30°C the

system operates at 30% of the maximum electrical energy input. Find the highest outside temperature for which the house may be maintained inside at 20°C.

(13.7) Two identical bodies of constant heat capacity $C_p$ at temperatures $T_1$ and $T_2$ respectively are used as reservoirs for a heat engine. If the bodies remain at constant pressure, show that the amount of work obtainable is

$$W = C_p \left(T_1 + T_2 - 2T_\mathrm{f}\right), \tag{13.37}$$

where $T_\mathrm{f}$ is the final temperature attained by both bodies. Show that if the most efficient engine is used, then $T_\mathrm{f}^2 = T_1 T_2$.

(13.8) A building is maintained at a temperature $T$ by means of an ideal heat pump which uses a river at temperature $T_0$ as a source of heat. The heat pump consumes power $W$, and the building loses heat to its surroundings at a rate $\alpha(T - T_0)$, where $\alpha$ is a positive constant. Show that $T$ is given by

$$T = T_0 + \frac{W}{2\alpha}\left(1 + \sqrt{1 + 4\alpha T_0/W}\right). \tag{13.38}$$

(13.9) Three identical bodies of constant thermal capacity are at temperatures 300 K, 300 K and 100 K. If no work or heat is supplied from outside, what is the highest temperature to which any one of these bodies can be raised by the operation of heat engines? If you set this problem up correctly you may have to solve a cubic equation. This looks hard to solve but in fact you can deduce one of the roots [hint: what is the highest temperature of the bodies if you do nothing to connect them?].

## Sadi Carnot (1796–1832)

Sadi Carnot's father, Lazare Carnot (1753–1823), was an engineer and mathematician who founded the École Polytechnique in Paris, was briefly Napoleon Bonaparte's minister of war and served as his military governor of Antwerp. After Napoleon's defeat, Lazare Carnot was forced into exile. He fled to Warsaw in 1815 and then moved to Magdeburg in Germany in 1816. It was there in 1818 that he saw a steam engine, and both he and his son Sadi Carnot, who visited him there in 1821, became hooked on the problem of understanding how it worked.

**Fig. 13.12** Sadi Carnot

Sadi Carnot had been educated as a child by his father. In 1812 he entered the École Polytechnique and studied with Poisson and Ampère. He then moved to Metz and studied military engineering, worked for a while as a military engineer, and then moved back to Paris in 1819. There he became interested in a variety of industrial problems as well as the theory of gases. He had now become skilled in tackling various problems, but it was his visit to Magdeburg that proved crucial in bringing him the problem that was to be his life's most important work. In this, his father's influence was a significant factor in the solution to the problem. Lazare Carnot had been obsessed by the operation of machines all his life and had been particularly interested in thinking about the operation of water-wheels. In a water-wheel, falling water can be made to produce useful mechanical work. The water falls from a reservoir of high potential energy to a reservoir of low potential energy, and on the way down, the water turns a wheel which then drives some useful machine such as a flour mill. Lazare Carnot had thought a great deal about how you could make such systems as efficient as possible and convert as much of the potential energy of the water as possible into useful work.

Sadi Carnot was struck by the analogy between such a water-wheel and a steam engine, in which heat (rather than water) flows from a reservoir at high temperature to a reservoir at low temperature. Carnot's genius was that rather than focus on the details of the steam engine he decided to consider an engine in abstracted form, focussing purely on the flow of heat between two thermal reservoirs. He idealized the workings of an engine as consisting of simple gas cycles (in what we now know as a Carnot cycle) and worked out its efficiency. He realised that to be as efficient as possible, the engine had to pass slowly through a series of equilibrium states and that it therefore had to be reversible. At any stage, you could reverse its operation and send it the other way around the cycle. He was then able to use this fact to prove that all reversible heat engines operating between two temperatures had the same efficiency.

This work was summarized in his paper on the subject, *Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance* (Reflections on the motive power of fire and machines fitted to develop That power) which was published in 1824. Carnot's paper was favourably reviewed, but had little immediate impact. Few could see the relevance of his work, or at least see past the abstract argument and the unfamiliar notions of idealized engine cycles; his introduction, in which he praised the technical superiority of English engine designers, may not have helped win his French audience. Carnot died in 1832 during a cholera epidemic, and most of his papers were destroyed (the standard precaution following a cholera fatality). The French physicist Émile Clapeyron later noticed his work and published his own paper on it in 1834. However, it was yet another decade before the work simultaneously came to the notice of a young German student, Rudolf Clausius, and a recent graduate of Cambridge University, William Thomson (later Lord Kelvin), who would each individually make much of Carnot's ideas. In particular, Clausius patched up and modernized Carnot's arguments (which had assumed the validity of the prevailing, but subsequently discredited, caloric theory of heat) and was motivated by Carnot's ideas to introduce the concept of entropy.