Advanced Microeconomics 2
Helsinki GSE
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Problem Set 4, Due December 13, 2023 (first three questions are from last year's final exam)

1. Answer the following short questions.

   (a) Two agents have preferences represented by utility functions $u_1(x)$ and $u_2(x)$ over allocations x in a finite set X. Assume that x^* is Pareto efficient if an allocation is chosen once. Suppose that an $x^j \in X$ is chosen separately in two periods $j \in \{1, 2\}$ and the utility function of agent i for (x^1, x^2) is given by $u_i(x^1) + u_i(x^2)$ (i.e. the utility is the sum of the utilities in each period). Prove or disprove that the allocation $x^1 = x^2 = x^*$ is Pareto efficient in the two-period allocation problem?

   (b) The following table lists the preferences of four agents $i \in \{1, 2, 3, 4\}$ represented by the columns over four houses $j \in \{a, b, c, d\}$ represented by the rows. If a house j is on a higher row than j' in column i, then $j \succ_i j'$.

1	2	3	4
a	a	c	b
c	d	(a)	(c)
(d)	(b)	b	d
b	c	d	a

Figure 1: Preferences over houses and the initial allocation circled

The initial allocation of houses in this economy is (d, b, a, c) (where agent i gets the i^{th} element in the vector as her house)? Define a competitive equilibrium for this economy and find a competitive equilibrium price and a competitive equilibrium allocation of houses.

2. Answer the following problems for an exchange economy.

   (a) State and prove the first welfare theorem for exchange economies.

   (b) Find the Pareto efficient allocations for a society consisting of two agents $i \in \{1, 2\}$ with utility functions

   $$u_1(x_{11}, x_{12}) = \ln(x_{11}) + \alpha x_{12},$$

   $$u_2(x_{21}, x_{22}) = \beta x_{21} + \ln(x_{22}).$$

   where $\alpha, \beta > 0$, and the total quantities of the goods are $\overline{x}_1 = 5, \overline{x}_2 = 3$.

   (c) Find a competitive equilibrium for an economy where the agents have utility functions as above with $\alpha = \beta = 1$, and their initial endowments are: $\omega_1 = (1, 2), \omega_2 = (4, 1)$.

3. Consider the following economy: There are two periods, two states in the second period, and two consumers. There is one physical commodity. The endowment in period 0 is 2 for both consumers. Consumer 1 has an endowment $\omega_{1s} = 1$ in both states in period 1 and consumer 2 has an endowment $\omega_{2s} = 2$ in both states in period 1. By x_0 we denote the consumption in period 0. By x_{1s} we denote the consumption in period 1 in state s. The utility functions of the consumers are given by:

   $$u^i(x_0, x_{11}, x_{12}) = \ln x_0 + \sum_{s=1}^{2} p_s \ln x_{1s}.$$

   (a) Suppose that the agents can save their period 0 endowment so that if i saves y_i in period 0, she can consume $\omega_{is} + y_i$ in state s in period 1. Find the optimal savings for the two consumers.

   (b) Assume that the good is perishable and cannot be stored between periods. The agents have access to an asset that pays 1 in both states in period 1 and costs q in period 0. The asset is in zero net supply so that market clearing requires $z_1 + z_2 = 0$, where z_i is the asset demand of agent i. If $z_i < 0$, then i receives qz_i in period 0 and pays z_i in both states in period 1. Define a competitive equilibrium for this model and solve the equilibrium asset price q and the equilibrium asset demand and consumption allocation.

   (c) Continue assuming that the good is perishable and solve for the competitive equilibrium when the only asset available to the agents pays 1 unit in state 1 and nothing in state 2.

4. An economy consists of a continuum of mass 2 of agents and a continuum of mass 3 of houses. The agents' willingness to pay for housing quality v_i is uniformly distributed on $[0, 2]$ so that the utility for an agent of type v_i from owning a house of quality q_j is $v_i q_j$. The houses have a uniform quality distribution on $[0, 3]$.

   (a) The houses are owned by absentee landlords and their opportunity cost of renting (outside option for holding the house of quality q_j) is $\gamma + \alpha q_j$. Assume that the agents and the house owners have quasilinear utilities in utility from housing and money. Find the efficient allocation of houses (determine also which houses stay with the absentee landlords).

   (b) Assume that $\gamma = \alpha = 1$ and solve for the competitive equilibrium price (function) $p(q)$ for the houses and describe the equilibrium allocation of houses.

5. Consider a large population of identical children. A child's preferences can be represented by a utility function $U : \mathbb{R} \longrightarrow \mathbb{R}$, which takes a number of gifts received as input. Furthermore, $U' > 0$ and $U'' < 0$.

   Christmas is coming. There are two possible states of Christmas. With a probability of $\frac{1}{2}$, Christmas is happy, in which case each child receives two Christmas gifts from Santa Claus. With the remaining probability of $\frac{1}{2}$, Christmas is miserable, in which case each child receives only one Christmas gift from Santa Claus. The state of Christmas is realized on Christmas Eve.

   Children can trade three different assets. A child who holds one *risk-free asset* on Christmas Eve is entitled to one additional Christmas gift regardless of the state of Christmas. A holder of a *procyclical asset* is entitled to 2 additional Christmas gifts if Christmas is happy, and to 0 additional gifts if Christmas is miserable. A child who holds a *countercyclical asset* on Christmas Eve is entitled to 2 additional Christmas gifts if Christmas is miserable, and to 0 additional gifts if Christmas is happy. Denote the pre-Christmas prices of these assets by P^{RF}, P^{PC}, and P^{CC}, respectively (the prices are measured in terms of gifts). Short sales are allowed, and the assets must be in zero net supply.

   Hint: Notice that in the equilibrium, there is no asset trade, and each child is indifferent between buying and selling any of the assets before Christmas.

(a) Determine the following three price ratios: P^{RF}/P^{PC}, P^{PC}/P^{CC}, and P^{RF}/P^{CC}. (The answer should be in terms of $U'(1)$ and $U'(2)$.) Which one of the assets is the most expensive one, and which asset is the cheapest one? Which asset has the highest expected rate of return?

(b) Consider an otherwise similar situation, but with a small difference regarding the miserable state of Christmas: if Christmas is miserable, half of the children receive 0 Christmas gifts from Santa Claus, whereas the remaining half of the children receive 2 gifts from Santa Claus. The children are still identical *ex ante*. Determine the price ratio P^{PC}/P^{CC} in this situation. If $U''' = 0$ (in the relevant part of the domain), is P^{PC}/P^{CC} equal, greater, or less than in Part (a)? If $U''' > 0$, is P^{PC}/P^{CC} equal, greater, or less than in Part (a)?