



Aalto University
School of Science
and Technology

CS-E5745 Mathematical Methods for Network Science

Mikko Kivelä

Department of Computer Science
Aalto University, School of Science
mikko.kivela@aalto.fi

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ERGMs and SBMs

- ▶ Learning goals this week:
 - ▶ Learn the basics of exponential random graphs (ERGMs)
 - ▶ Learn the basics of stochastic block models (SBMs)
- ▶ Materials: Newman 15.2

Graph ensembles with given properties

- ▶ Ensembles where graphs have predetermined values for properties $x(G) = x^*$.
- ▶ **“Microcanonical”**: G in the ensemble iff $x(G) = x^*$.
 - ▶ Otherwise maximally random: $P(G) = 1/c$ if $x(G) = x^*$ and $P(G) = 0$ if $x(G) \neq x^*$
 - ▶ Difficult to deal with analytically
- ▶ **“(Macro)canonical”**: $\langle x \rangle = x^*$.
 - ▶ Otherwise maximally random: $\max_P [-\sum_G P(G) \log P(G)]$
 - ▶ Leads to “exponential random graphs” (ERGM)
 - ▶ Nice statistical properties
 - ▶ Depending on x , might be difficult or easy to deal with analytically

Exponential random graphs (ERGMs)

- ▶ Class of network models for which

$$P(G) = \frac{e^{-\sum_i x_i(G)\theta_i}}{Z(\theta)},$$

- ▶ where each x_i is an observation (a number) that we measure from the network

Exponential random graphs (ERGMs)

- + ERGMs are in the *exponential family* of distributions:
 - ▶ Desirable statistical properties
 - ▶ Maximum entropy derivation
- The normalisation constant Z can be difficult to calculate
 - ▶ Sampling from the model can be difficult
 - ▶ Fitting the model can be even more difficult

ERGM in the literature

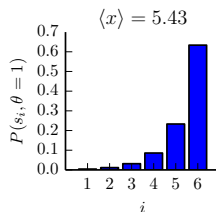
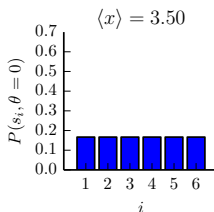
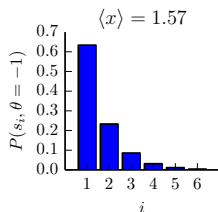
- ▶ You already know examples of ERGMs:
 - ▶ The (p version of) Erdős-Rényi networks
 - ▶ The “soft” configuration model
- ▶ Stochastic block models are ERGMs
- ▶ The social network analysis literature uses ERGMs extensively
 - ▶ Their models don't usually have a solution for Z
 - ▶ Selecting wrong observables x_i leads to computational problems
 - ▶ Selecting x_i is an art form by itself

ERGMs from the maximum entropy principle

- ▶ ERGMs are probability distributions of graphs for which:
 1. The expected value of each observable gets some predetermined value $\langle x_i(G) \rangle = x_i^*$, s.t.
 2. the entropy of the distribution is maximised.
- The most random probability distribution with a specified expected value
- ▶ If we only know the expected value of the observables, ERGM gives us the “best guess” of the distribution
 - ▶ “the least biased estimate possible”
 - ▶ “maximally noncommittal with regard to missing information”
- ▶ Proof as an exercise

A simple example of exponential distributions

- ▶ States of the system: $s \in \{s_1 \dots s_6\}$
- ▶ Observable: $x(s_i) = i$
- ▶ $P(s_i|\theta) = \frac{e^{i\theta}}{\sum_{j=1}^6 e^{j\theta}}$
- ▶ $P(s_i|0) = \frac{1}{6}$



ERGMs and statistics

- ▶ Part of the “exponential family” of distributions
 - ▶ Exponential distribution, normal distribution, . . .
- ▶ x is the vector of “sufficient statistics”
- ▶ If the model is defined without fixing parameters θ and you have a single observed network G_o
 - ▶ Choosing $\theta = \hat{\theta}$ such that $\langle x_i(G) \rangle = x_i(G_o)$ *equivalent to* finding the maximum-likelihood estimates
$$\hat{\theta} = \operatorname{argmax}_{\theta} P(G_o|\theta)$$
 - ▶ Proof as exercise.

ERGMs and statistical physics

- ▶ The ERGMs are of the same form as canonical ensembles, the Boltzmann distribution, ...
 - ▶ Distribution of energy levels of a system (at state S and observables x_i)
- ▶ Hamiltonian: $H = \sum_i x_i(S)\theta_i$
- ▶ Partition function: $Z(\theta)$
- ▶ Free energy: $F = -\ln Z$
- ▶ Chemical potentials, inverse temperature, ...: θ_i

ERGM version of the configuration model

- ▶ Observables: degree of each node $k_i = k_i(G)$
 - ▶ In our ensemble $\langle k_i \rangle = k_i^*$ (k_i^* are the target values)
- ▶ Our Hamiltonian is:

$$H(G, \theta) = \sum_i k_i(G) \theta_i \quad (1)$$

- ▶ So the distributions is:

$$P(G|\theta) = \frac{e^{-H(G, \theta)}}{Z(\theta)} = \frac{e^{-\sum_i k_i(G) \theta_i}}{Z(\theta)} \quad (2)$$

ERGM version of the configuration model

- ▶ The Hamiltonian can be written as:

$$H(G, \theta) = \sum_i \theta_i k_i = \sum_i \theta_i \sum_j A_{ij} = \sum_{i < j} (\theta_i + \theta_j) A_{ij}. \quad (3)$$

- ▶ In this case the partition function can be written without the sum over all graphs!

$$Z(\theta) = \sum_{G \in \mathcal{G}} e^{-H(G, \theta)} = \dots = \prod_{i < j} (1 + e^{-(\theta_i + \theta_j)}). \quad (4)$$

- ▶ Similar derivation as an exercise.

ERGM version of the configuration model

- ▶ In total the factors can be reorganised in a way that:

$$P(G|\theta) = \prod_{i < j} p_{ij}^{A_{ij}} (1 - p_{ij})^{1 - A_{ij}}, \quad (5)$$

where the model parameters have been transformed s.t.

$$p_{ij} = \frac{1}{1 + e^{\theta_i + \theta_j}}. \quad (6)$$

- ▶ When we require that $\langle k_i \rangle = k_i^*$, we need to solve θ_i from

$$k_i^* = \sum_j p_{ij} = \sum_j \frac{1}{1 + e^{\theta_i + \theta_j}}, \quad \forall i \quad (7)$$

ERGM version of the configuration model

- ▶ In the “sparse limit”, where $e^{\theta_i+\theta_j} \gg 1$ we can write

$$k_i^* = \sum_j \frac{1}{1 + e^{\theta_i+\theta_j}} \approx \sum_j e^{-\theta_i} e^{-\theta_j}. \quad (8)$$

- ▶ Solution:

$$e^{-\theta_i} \approx \frac{k_i^*}{\sqrt{2m}}$$
$$p_{ij} \approx e^{-\theta_i} e^{-\theta_j} = \frac{k_i^* k_j^*}{2m}$$

- ▶ This is the “soft” configuration model from the first lecture!
- ▶ The sparse limit approximation can be written $1/p_{ij} \gg 1$

About the partition function

- ▶ In the configuration model we could write $Z(\theta)$ without the sum over all graphs
 - ▶ One can always do it IF the Hamiltonian can be written in form $H = \sum_{ij} \Theta_{ij} A_{ij}$
 - ▶ This doesn't always happen!
- ▶ It is difficult to do calculations if $Z(\theta)$ cannot be solved
 - ▶ MCMC methods for sampling and inference

Stochastic block model (SBM)

- ▶ Each node i belongs to block $b_i \in \{1, \dots, K\}$
- ▶ Links with probability depending on their blocks p_{rs} (prob. of link between block r and s)

$$P(G|b, \{p_{rs}\}) = \prod_{i < j} p_{b_i b_j}^{A_{ij}} (1 - p_{b_i b_j})^{1 - A_{ij}} . \quad (9)$$

- ▶ p_{sr} is sometimes called the “block matrix”
 - ▶ One can think of it spanning a new more simple “block network”

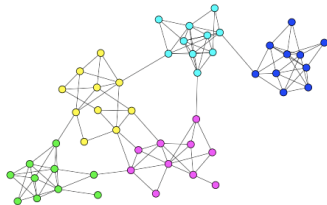
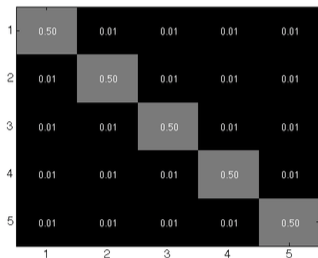
SBM as ERGM

- ▶ SBM is an ERGM!
- ▶ The observations are the number of links between blocks r and s : e_{rs}
- ▶ The Z can be solved and the form in the previous slide is returned with change of variables

$$p_{rs} = \frac{1}{1 + e^{\lambda_{rs}}} \quad (10)$$

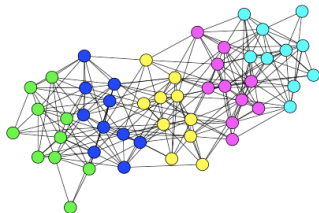
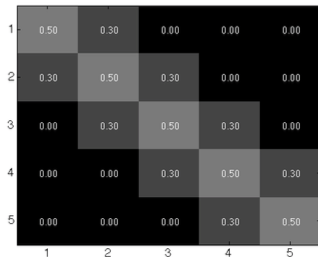
- ▶ Derivation as an exercise

SBM examples¹ (1/4)

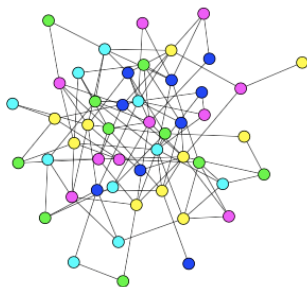
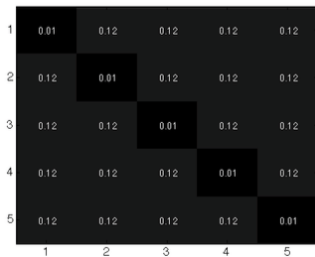


¹<http://tuvalu.santafe.edu/~aaronc/courses/5352/>

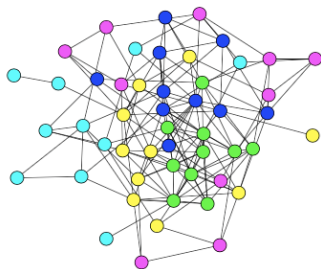
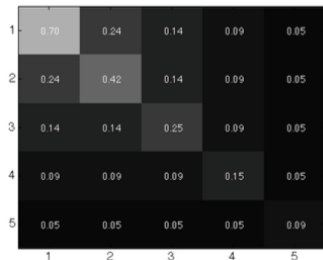
SBM examples (2/4)



SBM examples (3/4)



SBM examples (4/4)

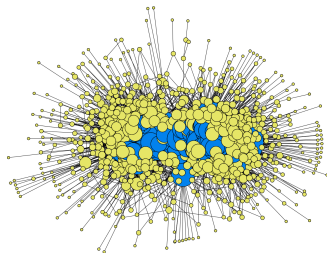
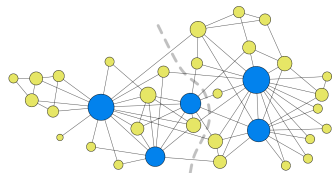


Inference with SBM

- ▶ SBM produces a network with the planted partition b_i and the block matrix p_{rs}
- ▶ Inference: we want to know the most likely model to produce the data
- ▶ Finding p_{rs} is easy given a network G and b_i (exercise)
- ▶ Finding b_i difficult \rightarrow heuristic algorithms

Problem with SBM: degree distributions

- ▶ Real networks have fat-tail degree distributions, SBM finds this structure²



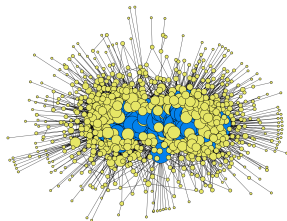
²Karrer & Newman, PRE 83, 016107

Degree-corrected SBM

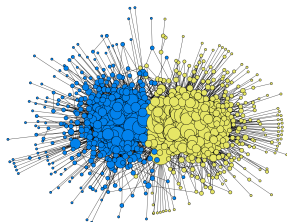
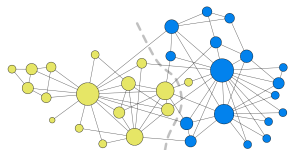
- ▶ Idea: combine the ERGM configuration model and SBM
- ▶ Observables: the degrees of nodes AND number of links between blocks
 - ▶ Model parameters related to degree θ_i and blocks λ_{rs}
 - ▶ The best fit to data explains degrees with θ and blocks with λ_{rs}

Problem with SBM: degree distributions

No degree correction:

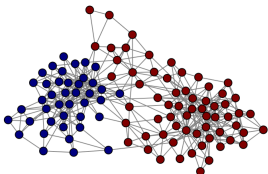


With degree correction:

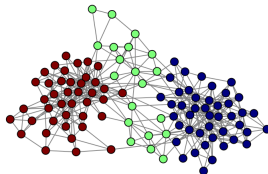


Problem with SBM: overfitting

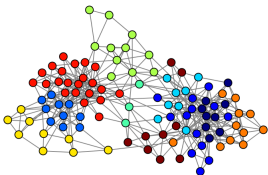
- ▶ More blocks \rightarrow better likelihood³



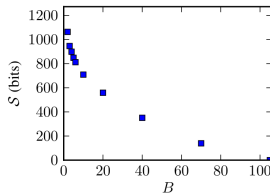
$B = 2$



$B = 3$



$B = 10$



$$S = -\log P(G|\theta), \quad B \text{ number of blocks}$$

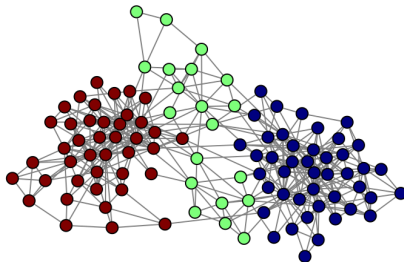
³Figures from Peixoto, Como'16

Minimum description length and SBM

- ▶ Instead of maximising (log) likelihood $P(G|\theta)$ maximise the posterior $P(\theta|G) = \frac{P(G|\theta)P(\theta)}{P(G)}$
- Minimise: $-\ln P(\theta|G) = -\ln P(G|\theta) - \ln P(\theta) + \ln(P(G))$,
 - ▶ $P(G)$ is constant
 - ▶ $S = -\ln P(G|\theta)$: information needed to describe G when model known
 - ▶ $L = -\ln P(\theta)$: information needed describe to the model
 - ▶ Description length $S + L$
- ▶ Calculating L based on giving each partition b equal probability (uniform prior) etc.

Minimum description length and SBM

- ▶ MDL finds a compromise between the model fit S and complexity of the model L^4



$$B = 3$$

⁴Peixoto, PRL 110, 148701 (2013)

ERGMs in social network analysis (SNA)

- ▶ ERGMs are a popular tool for analysing small social networks
 1. select the observables x_i (“network statistics”) based on a research question (often includes metadata on nodes),
 2. fit the model to data, and
 3. look at the θ_i to interpret the results
- ▶ The $Z(\theta)$ not solvable \rightarrow numerical methods to find MLE θ
 - ▶ Find numerically θ s.t. $\langle x_i \rangle = x_i^*$, with MCMC methods
 - ▶ Selecting wrong observables x_i might lead to serious computational problems (“degeneracy”: multiple parameter combinations might explain the data)
 - ▶ Often p-values are calculated for testing a null-model where $\theta_i = 0$

Example: ERGMs in SNA

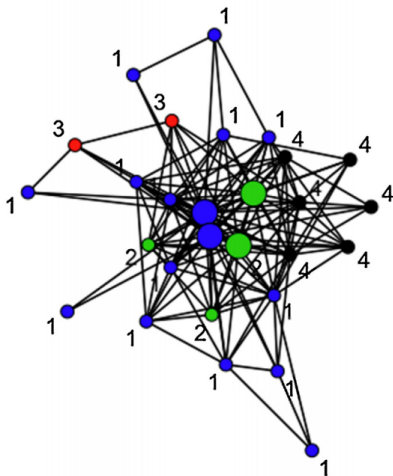
- Data: 6+6 classes, around 24 nodes per class⁵

	"3-year-olds"			"4-year-olds"		
	$\hat{\mu}_{WLS}$	SE	σ^2	$\hat{\mu}_{WLS}$	SE	σ^2
Reciprocity	4.61**	.28	.00	4.59**	.33	.26
Alternating k-instar	1.10**	.22	.02	1.12**	.20	.00
Alternating k-outstar	-1.75**	.09	2.82**	-1.12**	.48	.35
Alternating k-triangle t	.30**	.08	.02*	.21**	.04	.00
Alternating k-two-path	-.51**	.10	.03	-.60**	.07	.01
Ego sex (σ^x)	-.63**	.21	.00	-.54**	.21	.00
Alter sex (σ^y)	-.27	.16	.00	-.13	.15	.01
Sex similarity	.87**	.15	.00	1.21**	.17	.00

⁵Daniel et al., Social Net. 35(1), 25 (2013)

Example: ERGMs in SNA

- ▶ Social network of judges⁶



⁶Lazega et al., Social Net. 48, 10 (2017)

Example: ERGMs in SNA

Effects	Parameter estimate	Standard error
<i>Variables of interest</i>		
Judges apply the same rule	-0.579	0.272
Judges belong to same capitalism block	-0.707	0.452
Judges apply the same rule AND belong to continental Europe capitalism block	1.242	0.346
Judges apply the same rule AND belong to UK capitalism block	0.673	0.335
Judges apply the same rule AND belong to Scandinavia capitalism block	0.951	0.402
Judges apply the same rule AND belong to southern Europe capitalism block	0.945	0.351
<i>Endogenous network controls</i>		
Density	-4.537	1.039
Reciprocity	1.261	0.394
Indegree control 1 (Markov)	0.012	0.001
Outdegree control 1 (Markov)	0.012	0.001
Twopath	-0.087	0.025
Indegree control 2	-0.061	0.331
Outdegree control 2	-0.340	0.350
Transitive closure	1.167	0.211
Cyclic closure	0.029	0.120
Transitive connectivity	-0.058	0.032