Appendix A

Extended half line

The extended half-line is a set $[0, \infty] = \mathbb{R}_+ \cup \{\infty\}$, where $\mathbb{R}_+$ denotes the nonnegative real numbers and ∞ is an element not in $\mathbb{R}_+$. The extended half-line has a natural algebraic, order-theoretic, and topological structures. The topology then induced a natural Borel sigma-algebra to this space.

A.1 Algebraic structure

The sum and product operations on $\mathbb{R}_+$ are extended to $[0, \infty]$ by defining

$$x + \infty \;=\; \infty + x \;=\; \infty \qquad \text{for } x \geq 0,$$

and

$$x \cdot \infty \;=\; \infty \cdot x \;=\; \begin{cases} 0 & \text{for } x = 0, \\ \infty & \text{for } x > 0. \end{cases}$$

The set $[0, \infty]$ equipped with these operations is a semi-ring[1] with additive identity 0 and multiplicative identity 1.

A.2 Order

We define a relation $\leq$ on $[0, \infty]$ by saying that $x \leq y$ if either $y = \infty$, or $x, y \in \mathbb{R}_+$ and $x \leq y$ in the usual ordering on the real line. We denote $x < y$ whenever $x \leq y$ and $x \neq y$. Then set $[0, \infty]$ then becomes totally ordered[2], and a complete lattice in the sense that $\inf(A), \sup(A) \in [0, \infty]$ for every

[1] https://en.wikipedia.org/wiki/Semiring

[2] A partial order is a relation $\leq$ that is reflexive ($x \leq x$), antisymmetric ($x \leq y$, $y \leq x \implies x = y$), and (transitive $x \leq y$, $y \leq z \implies x \leq z$). A total order is a partial order such for every x, y, either $x \leq y$ or $y \leq x$.

nonempty $A \subset [0,\infty]$. We denote intervals with endpoints $a, b \in [0,\infty]$ by (a,b), $(a,b]$, $[a,b)$, and $[a,b]$ as usual.

A.3 Topology

Sets of the form $(a,b) = \{x : a < x < b\}$ are called open intervals in $[0,\infty]$. Sets of the form $[0,a) = \{x : x < a\}$ and $(a,\infty] = \{x : x > a\}$ are called open rays in $[0,\infty]$. A set $A \subset [0,\infty]$ is called open if it can be expressed as a union[3] of open intervals and open rays in $[0,\infty]$. The collection of all open sets is denoted $\mathcal{T}([0,\infty])$ and called the topology[4] of $[0,\infty]$. Examples of open sets are $[0,\infty)$, $[0,\infty]$. Examples of closed sets include the singleton sets $\{a\}$ with $a \in [0,\infty]$ and the sets $[0,a]$ and $[0,\infty]$. This type of topology can be defined for any totally ordered space — in general such topologies are called order topologies.

One may verify that $F\colon [0,\infty] \to [0,1]$ defined by

$$F(x) \;=\; \begin{cases} 1 - e^{-x}, & 0 \le x < \infty, \\ 1, & x = \infty \end{cases}$$

is an increasing and continuous bijection with an increasing and continuous inverse

$$F^{-1}(x) \;=\; \begin{cases} \log \frac{1}{1-x}, & 0 \le x < 1, \\ \infty, & x = 1. \end{cases}$$

Therefore F serves as an order isomorphism and a topology isomorphism (homeomorphism) between $[0,\infty]$ and $[0,1]$. Hence these sets share the same order-theoretic and topological properties. Especially, we find that $[0,\infty]$ is a compact and connected topological space. We can express the topology of the extended half line as $\mathcal{T}([0,\infty]) = F^{-1}(\mathcal{T}([0,1]))$.

Proposition A.3.1. *Every open set in $[0,\infty]$ can be expressed as a countable union of open intervals and open rays.*

Proof. Denote by $\mathbb{Q}_+$ the set of nonnegative rational numbers. Denote by $\mathcal{R}$ the set family consisting of open intervals (a,b), lower open rays $[0,a)$, and upper open rays $(a,\infty]$ with *rational* endpoints $a, b \in \mathbb{Q}_+ \cup \{\infty\}$. We will show that every open set in $[0,\infty]$ can be expressed as a union of sets in $\mathcal{R}$.

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[3] Also, the empty set is defined to be an open set.

[4] In general, a topology is a set family on Ω that contains $\emptyset, \Omega$ and is closed under arbitrary unions and finite intersections.

Let $U \subset [0, \infty]$ be open. Pick a point $x \in U$. Then x is either contained in an open interval (a, b), in a lower open ray $[0, a)$, or in an upper open ray $(a, \infty]$, for some $a, b \in [0, \infty]$.

(i) If $x \in (a, b)$, then $x \in (a', b')$ for some $a', b' \in \mathbb{Q}_+ \cup \{\infty\}$.

(ii) If $x \in [0, a)$, then $x \in [0, a')$ for some $a' \in \mathbb{Q}_+ \cup \{\infty\}$.

(iii) If $x \in (a, \infty]$, then $x \in (a', \infty]$ for some $a' \in \mathbb{Q}_+ \cup \{\infty\}$.

To every point $x \in U$ we may hence associate a set $R_x \in \mathcal{R}$ that contains x. As a consequence, $U = \cup_{x \in U} R_x$ can be expressed as a union of sets in $\mathcal{R}$.

Finally, we note that the lower and upper open rays in $\mathcal{R}$ may be indexed using the elements of $\mathbb{Q}_+ \cup \{\infty\}$, and the open intervals in $\mathcal{R}$ may be indexed using pairs in $(\mathbb{Q}_+ \cup \{\infty\})^2$. Because the sets $\mathbb{Q}_+ \cup \{\infty\}$ and $(\mathbb{Q}_+ \cup \{\infty\})^2$ are countable, it follows that every open set in $[0, \infty]$ can be expressed as a countable union of sets in $\mathcal{R}$. In other words, $\mathcal{R}$ forms a countable basis for the topology of $[0, \infty]$. □

A.4 Borel sigma-algebra

The Borel sigma-algebra on $[0, \infty]$ is defined as $\mathcal{B}([0, \infty]) = \sigma(\mathcal{T}([0, \infty]))$, the smallest sigma-algebra containing the open sets of $[0, \infty]$.

Proposition A.4.1. *The family of closed lower rays $\mathcal{I} = \{[0, x] : x \in \mathbb{R}_+\}$ is a π-system on $[0, \infty]$ that generates the Borel sigma-algebra $\mathcal{B}([0, \infty])$.*

Proof. The fact that $\mathcal{I}$ is a π-system follows immediately by noting that $[0, x] \cap [0, y] = [0, x \wedge y]$ for all $x, y \in \mathbb{R}_+$. To finish the proof, it suffices to verify that

$$\mathcal{I} \subset \sigma(\mathcal{T}) \tag{A.4.1}$$

and

$$\mathcal{T} \subset \sigma(\mathcal{I}), \tag{A.4.2}$$

where $\mathcal{T} = \mathcal{T}([0, \infty])$ is the set family of open sets in $[0, \infty]$.

Verifying (A.4.1) is easy because each closed ray $[0, x]$ is the complement of an open ray $(x, \infty]$, and therefore $[0, x]$ belongs to $\sigma(\mathcal{T})$. To verify (A.4.1), we proceed in three steps.

(i) First we observe that $(a, b] \in \sigma(\mathcal{I})$ for all $a, b \in [0, \infty]$, because $(a, b] = [0, b] \cap [0, a]^c$: when $b < \infty$, both $[0, a]$ and $[0, b]$ belong to $\mathcal{I}$; when $b = \infty$, $(a, b] = [0, a]^c$ is the complement of a set in $\mathcal{I}$.

(ii) By applying (i), we see that $(a,b) \in \sigma(\mathcal{I})$ for all $a,b \in [0,\infty]$, because

$$(a,b) = \begin{cases} \cup_{n\in\mathbb{N}}(a, b-\frac{1}{n}], & b < +\infty, \\ \cup_{n\in\mathbb{N}}(a,n], & b = +\infty. \end{cases}$$

(iii) By applying (ii), we see that $[a,b) \in \sigma(\mathcal{I})$ for all $a,b \in [0,\infty]$, because

$$\begin{aligned} [a,b) &= [0,b) \cap [0,a)^c \\ &= \Big([0,0] \cup (0,b)\Big) \cap \Big([0,0] \cup (0,a)\Big)^c. \end{aligned}$$

The claim $\mathcal{T} \subset \sigma(\mathcal{I})$ follows from the above observations, because every open set in $\mathcal{T}$ can be expressed as a countable union (see Proposition A.3.1) of intervals of form (a,b) and $[0,a)$ and $(a,\infty]$ with $a,b \in [0,\infty]$. □

A.5 Measurable functions

Let $(S,\mathcal{S})$ be a measurable space. A function $f\colon S \to [0,\infty]$ is called measurable if it is $\mathcal{S}/\mathcal{B}([0,\infty])$-measurable.

Proposition A.5.1. *f is measurable iff $\{s : f(s) \le x\} \in \mathcal{S}$ for all $x \in \mathbb{R}_+$.*

Proof. By Proposition A.4.1, the set family $\{[0,x]\colon x \in \mathbb{R}_+\}$, is a pi-system that generates $\mathcal{B}([0,\infty])$. The claim follows by Proposition 2.4.1. □