ELEC-E8116 Model-based control systems
Full exam 12. 12. 2023 / Solutions

1. Explain briefly the following concepts in control theory (shortly, only what they are and what they mean from control viewpoint)

- SVD (2 p.)
- LQ (2 p.)
- IMC (2 p.)

2.a. Consider a MIMO system. Draw a schema of the "one-degrees-of-freedom" control configuration. Define the concepts *loop transfer function*, *sensitivity function* and *complementary sensitivity function* for it. (3 p.)

2.b. Consider a SISO-case. Determine the region in the complex plane where $|S| = 1/\sqrt{2}$. Then determine the region where $|T| = 1/\sqrt{2}$. Do these regions have common points? If they do, what are these points? Interpretation from control perspective? (3 p.)

3. a. Explain briefly the following concepts

- Principle of Optimality (1 p.)
- Dynamic programming (1 p.)
- Waterbed effect (1 p.)

3. b. State the "Push through rule" and prove it. Remember to give the general matrix dimensions involved. (3 p.)

4. a. Explain the *Receding Horizon Principle* in Model Predictive Control. (2 p.)

4. b. Explain shortly what is meant by the Relative Gain Array (RGA) and what is its meaning in control engineering. (2 p.)

4.c. Consider a linear SISO system. Explain shortly what different definitions there exist for the concept *bandwidth*. Explain these shortly. How can they be characterized in terms of control performance? (2 p.)

5. Consider the system

$$\dot{x}_1(t) = -x_1(t) + u(t)$$
$$\dot{x}_2(t) = x_1(t)$$

The criterion to be minimized is

$$J = \int_0^\infty (x_2^2(t) + 0.1u^2(t))dt$$

Determine the optimal control law and the optimal cost. (6 p.)

Hint. Remember that the solution matrix of the Riccati equation is symmetric and positive definite.

Some formulas that might be useful:

$$\dot{x} = Ax + Bu, \quad t \geq t_0$$

$$J(t_0) = \frac{1}{2} x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x^T Qx + u^T Ru \right) dt$$

$$S(t_f) \geq 0, \quad Q \geq 0, \quad R > 0$$

$$-\dot{S}(t) = A^T S + SA - SBR^{-1}B^T S + Q, \quad t \leq T, \quad \text{boundary condition } S(t_f)$$

$$K = R^{-1}B^T S$$

$$u = -Kx\,, \quad J^*(t_0) = \frac{1}{2} x^T(t_0)S(t_0)x(t_0)$$

$$\int_0^\infty \log\, |S(i\omega)|\, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\ (\mathrm{p_i})$$

$$|W_T(p_1)| \leq 1 \quad \Rightarrow \quad \omega_0 \geq \frac{p_1}{1 - 1/T_0}$$

$$|W_S(z)| \leq 1 \quad \Rightarrow \quad \omega_0 \leq \left(1 - 1/S_0\right) z$$

Solutions:

1.

SVD: Every *n x m* matrix *A* (real or complex-valued) has the *singular value decomposition* (SVD)

$$A = U\Sigma V^*$$

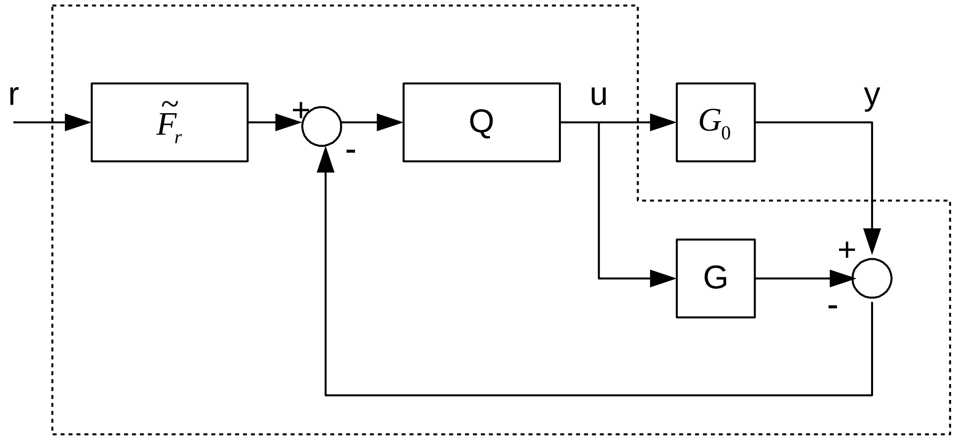
where dim(*U*) = *nxn*, dim(V) = *mxm* and dim(Σ) = *nxm*. The matrices *U* and *V* are *unitary* ($U^*U = I, V^*V = I$) and Σ is a *real-valued* matrix containing the non-negative *singular* values of matrix *A* in its main diagonal. The *gain* of the matrix is between the maximum and minimum singular values, which occur at the *output directions* given by the columns of matrix *U* and *input directions* given by the columns of matrix *V*. That information can be used in multivariable control theory, where the matrix *A* is the frequency-dependent transfer function matrix of a MIMO system.

LQ: Optimal control with *Linear System* and *Quadratic Criterion*.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

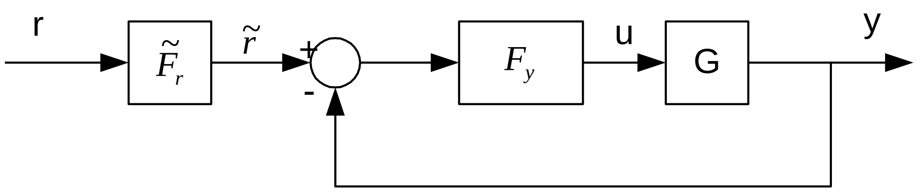
$$J = x^T(t_f)S(t_f)x(t_f) + \int_{t_0}^{t_f} [x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)]\, d\tau$$

IMC: *Internal model control*, where the model of the process is an essential component on the control structure. The topology of the control system in IMC is



where G denotes the process model. The controller is parameterized by the transfer functions Q and pre-compensator $\tilde{F}_r$. For details, see lecture slides, Chapter 6.

2a. For example, the below figure shows a one-degree-of-freedom control structure. (The pre-compensator can be left out and the controller F_y is often denoted by K in the literature.)



The loop transfer function L, sensitivity function S and complementary sensitivity function T are given as follows

$$L(j\omega) = G(j\omega)F_y(j\omega)$$
$$S(j\omega) = (I + L(j\omega))^{-1}$$
$$T(j\omega) = (I + L(j\omega))^{-1}L(j\omega)$$

2b.

$$S(j\omega) = \frac{1}{1+L(j\omega)} = \frac{1}{1+x+jy}$$

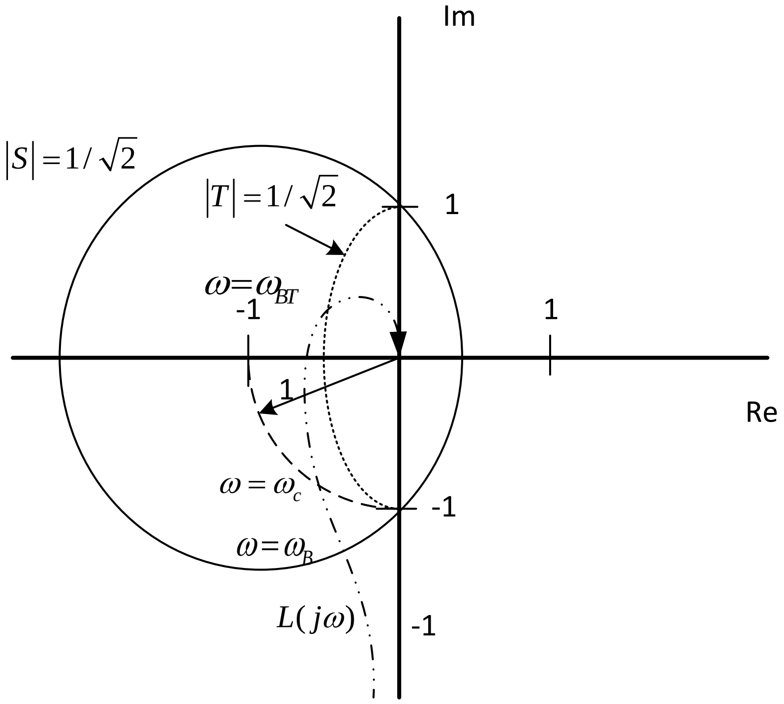
$$\Rightarrow |S|^2 = \frac{1}{(1+x)^2+y^2} = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} \Rightarrow (1+x)^2 + y^2 = 2$$

$$T(j\omega) = \frac{L(j\omega)}{1+L(j\omega)} = \frac{x+jy}{1+x+jy}$$

$$\Rightarrow |T|^2 = \left|\frac{L(j\omega)}{1+L(j\omega)}\right|^2 = \frac{|L|^2}{|1+L|^2} = \frac{x^2+y^2}{(1+x)^2+y^2} = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$$

$$\Rightarrow \ldots \Rightarrow (x-1)^2 + y^2 = 2$$

The magnitudes of both sensitivity functions are circles on the complex plane. The center points are (-1,0) and (1,0), respectively. Both have the radius $\sqrt{2} \approx 1.4$.



The common points are observed easily from the figure. They can also be obtained from the equations

$$(1+x)^2 + y^2 = 2$$
$$(x-1)^2 + y^2 = 2$$

Subtracting the second equation from the first one (or vice versa) it is easily deduced that the common solutions are $x = 0,\ y = \pm 1$, which agree with the figure.

Control perspective: If the Nyquist curve of the loop transfer function crosses the point $x = 0,\ y = -1$ or $x = 0,\ y = +1$ then the absolute value of the loop transfer function is 1 and the absolute values of the sensitivity functions are $1/\sqrt{2}$ as stated in the problem. There is not necessarily anything special about this. But if these gains are met *for the first time* in the Nyquist plot, then the angular frequencies $\omega_c, \omega_B, \omega_{BT}$ coincide. The different *bandwidths* then coincide as well.

(Not required): The figure also demonstrates the fact that under a mild condition (phase margin less than 90 degrees) it holds $\omega_B < \omega_c < \omega_{BT}$, where ω_B denotes the bandwidth, where L crosses $|S| = 1/\sqrt{2} \approx -3$ dB from below, ω_c is the gain crossover frequency $|L| = 1$ and ω_{BT} denotes the bandwidth, where L crosses $|T| = 1/\sqrt{2} \approx -3$ dB from above.

3a.

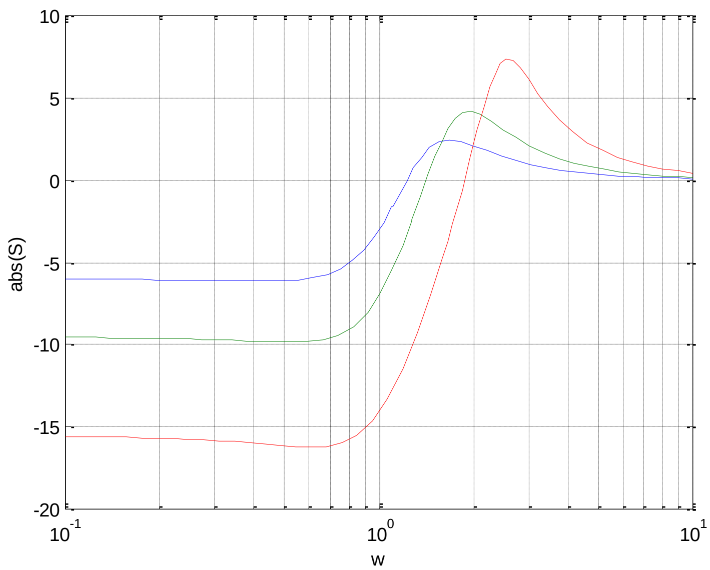
Principle of Optimality: The optimal trajectory has the property that irrespective of how a state in it has been reached, the remaining controls must be optimal from there on.

Dynamic Programming is a practical way of constructing optimal solutions by applying the Principle of Optimality. That leads to "calculation backwards in time". Starting from the end state we go back one time step and determine the possible states. From there the optimal control (first control) is determined and the optimal cost to the end is recorded. Then we go again one time step back. From each possible state the optimal route is determined by evaluating all possible first controls, seeing where they lead. Then the minimal first control is determined by minimizing the first controls plus the optimal cost after that (which was known). The first control and the optimal cost are recorded. The procedure continues to the beginning (initial state).

Waterbed effect. It comes from the function theoretic formula

$$\int_0^\infty \log |S(i\omega)| \, d\omega = \pi \sum_{i=1}^{M} \mathrm{Re}\,(\mathrm{p_i})$$

where S is the sensitivity function and p_i are RHP poles of the loop transfer function.



The integral states that if we apply a controller that "pushes down" the sensitivity function, it has to go up at some other frequency, because the integral (area) is a constant (zero, if there are no RHP poles in L). The name waterbed formula comes from that.

3b. Let A and B be $n \times m$ and $m \times n$ matrices, respectively. The push-through rule states that

$$A(I_{mxn} + BA)^{-1} = (I_{nxn} + AB)^{-1} A$$

where it has been assumed that the inverse matrices exist. A "direct" proof can be obtained by starting from the left hand side

$$\begin{aligned} &A(I+BA)^{-1} = (I+AB)^{-1}(I+AB)A(I+BA)^{-1} \\ &= (I+AB)^{-1}(A+ABA)(I+BA)^{-1} \\ &= (I+AB)^{-1}A(I+BA)(I+BA)^{-1} \\ &= (I+AB)^{-1}A \end{aligned}$$

An "indirect proof" is also possible (given in the exercises). There you start from the guessed result, push-through rule, and go through equivalences to an identity (like $I = I$). This is Ok, but you have to use equivalences to be able to go from the identity back to the claim. Be careful here. Note that if for matrices A and B it holds $A=B$, then for any compatible matrix Z

$$A = B \Rightarrow AZ = BZ$$

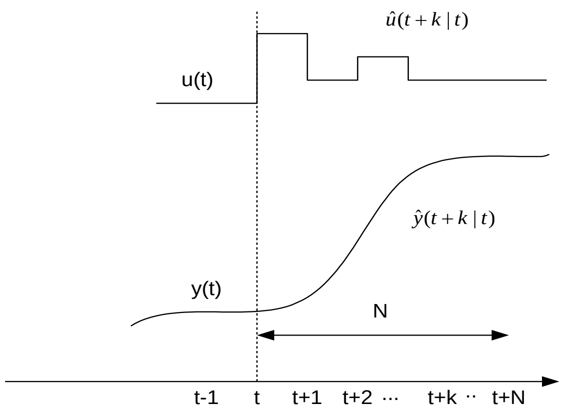
But in order that

$$A = B \Leftrightarrow AZ = BZ$$

would hold (equivalence, both ways must hold) the matrix Z has to be square matrix with full rank (=invertible).

The "direct" proof is more beautiful than the "indirect" one.

4a.



The Receding Horizon principle means that (see figure) at time t we estimate the state and output behaviour N_p steps ahead, where N_p is the *prediction horizon*. Then we minimize a cost function assuming N_c control moves, where N_c is the *control horizon*. Only the *first* control move is fed to the process input however. At the next time step the same procedure is repeated. The procedure can be regarded as a "window" that moves forward in time.

4b. In a multivariable system the Relative Gain Array measures the amount of couplings between the input and output channels.

$$\text{RGA}(A) = A.*\left(A^{-1}\right)^T$$

The coupling of input channel *i* to output channel *j* is "large", if the corresponding element in the RGA matrix is close to 1 (or (1,0) in the complex plane). The RGA analysis has mostly been used to construct decentralized controllers, where the independent control loops are determined based on RGA. RGA is a somewhat "heuristic" measure. Earlier it was used mostly in stationary analysis (s = 0), but nowadays variable frequencies, especially the bandwidth frequencies, are also considered important.

4c. Different definitions for bandwidth are for example ω_c (gain crossover angular frequency), ω_B (angular frequency where the sensitivity function reaches -3 dB from below), ω_{BT} (angular frequency, where the complementary sensitivity function reaches -3 dB from above). The bandwidth means generally the frequency band where control is effective, i.e. the closed loop performs well. See for example the closed loop formulas

$$z = G_c r + Sw - Tn + GS_u w_u$$
$$e = (I - G_c)r - Sw + Tn - GS_u w_u$$

where the control error e should be kept small. See also the answer to problem 2b, last part.

5. LQ problem with

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, R = 0.2$$

Note: $$J = \int_0^\infty (x_2^2(t) + 0.1u^2(t))dt = \frac{1}{2}\int_0^\infty (2x_2^2(t) + 0.2u^2(t))dt$$

The optimization horizon is infinite, and therefore the static Riccati equation applies ($\dot{S}(t) = 0$).

$$A^T S + SA - SBR^{-1}B^T S + Q = 0$$

Set $S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$ (note: symmetric) and write the Riccati equation. The following equations are obtained

$$-2s_{11} + 2s_{12} - 5s_{11}^2 = 0$$
$$\begin{cases} -s_{12} + s_{22} - 5s_{11}s_{12} = 0 \\ -s_{12} + s_{22} - 5s_{11}s_{12} = 0 \end{cases}$$
$$-5s_{12}^2 + 2 = 0$$

Starting from the last equation the whole Riccati equation can be solved analytically. Note that positive definite solution must be used (choose positive solutions in s_{ii}). After some nasty calculations we obtain

$$s_{11} = \frac{-2 + \sqrt{4 + 8\sqrt{10}}}{10} \approx 0.3413$$
$$s_{12} = s_{21} = \sqrt{\frac{2}{5}} \approx 0.6325$$
$$s_{22} = \sqrt{\frac{4 + 8\sqrt{10}}{10}} \approx 1.7117$$

The optimal control is

$$u^* = -R^{-1}B^T Sx = -5\begin{bmatrix} s_{11} & s_{12} \end{bmatrix} x \approx -\begin{bmatrix} 1.7064 & 3.1623 \end{bmatrix} x$$

and the optimal cost

$$J^*(0) = \frac{1}{2} x^T(0) S x(0) = x^T(0) \begin{bmatrix} \frac{s_{11}}{2} & \frac{s_{12}}{2} \\ \frac{s_{12}}{2} & \frac{s_{22}}{2} \end{bmatrix} x(0) \approx x^T(0) \begin{bmatrix} 0.1706 & 0.3162 \\ 0.3162 & 0.8558 \end{bmatrix} x(0)$$