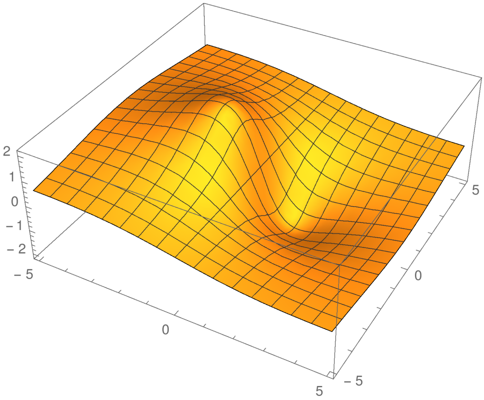
Welcome to

Differential and Integral Calculus 2
MS-A0211

What is new?

- functions of more than one variable
- 3D-thinking
- connecting geometric thinking to calculus



Why useful?

- real world problems are usually multivariable problems
  examples
  - profit of a company
  - volume of cylinder
  - kinetic energy
- or we like to investigate surfaces or curves
  examples
  - path of a particle
  - path of a robot hand
  - soap bubles

Lecture 1: Vector valued functions

Learning goals:

1. What are vector valued functions?
2. How limits, continuity and derivative are defined for vector valued functions?
3. What is a curve?
   - Kinematic point of view (velocity, acceleration, distance travelled)
   - Geometric point of view (tangent, arclength)

Where to find the material?

Corral 1.8, 1.9

Guichard 13.1, 13.2 and partially 13.3, 13.4

Active Calculus 9.5 - 9.8

Adams-Essex 12.1, 12.3

Vector valued functions

$$\mathbf{f}\colon I \to \mathbb{R}^n \quad \text{where } I \subset \mathbb{R} \text{ interval}$$

Example:

$$\mathbf{r}(t) = \begin{bmatrix} e^t \\ -4t \end{bmatrix} = e^t\mathbf{i} - 4t\mathbf{j} = (e^t, -4t)$$

Coordinate functions

$$\mathbf{f}\colon I \to \mathbb{R}^n \quad \text{where } I \subset \mathbb{R} \text{ interval}$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$$

- Functions $f_1, f_2, \ldots f_n$ are called coordinate functions
- Example:

  $$\mathbf{r}(t) = \begin{bmatrix} e^t \\ -4t \end{bmatrix} = e^t\mathbf{i} - 4t\mathbf{j} = (e^t, -4t)$$

  Coordinate functions $r_1(t) = e^t, r_2(t) = -4t$
- Typically we deal with funtions where $n = 2$ (or 3) then it is customary to denote them $f(t) = (x(t), y(t))$
  (or $f(t) = (x(t), y(t), z(t))$)

Limits, continuity and derivative

- $\lim_{t \to a} \mathbf{f}(t) = (\lim_{t \to a} f_1(t), \ldots, \lim_{t \to a} f_n(t))$
- vector valued function is continuous if all its coordinate functions are continuous
- $\mathbf{f}'(t) = \frac{d\mathbf{f}}{dt} = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h} =$
  $(\lim_{h \to 0} \frac{f_1(t+h) - f_1(t)}{h}, \ldots, \lim_{h \to 0} \frac{f_n(t+h) - f_n(t)}{h})$
  if the limit exists

What is a curve?

Continuous vector valued functions are called curves (or parametric presentations of curves)

- kinematic point of view - particle moving in a space
- geometric point of view - curve as a fixed set

Kinematic point of view

- natural interpretation: the function give the position of a "particle" at the time t
- **position:** $\mathbf{r}(t) = (x(t), y(t), z(t)) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$
- **average velocity:** $\dfrac{\triangle \mathbf{r}}{\triangle t} = \dfrac{\mathbf{r}(t+\triangle t) - \mathbf{r}(t)}{\triangle t}$
- **velocity:**
  $\mathbf{v}(t) = \dfrac{d}{dt}\mathbf{r}(t) = \lim\limits_{\triangle t \to 0} \dfrac{\mathbf{r}(t+\triangle t) - \mathbf{r}(t)}{\triangle t} = (x'(t), y'(t), z'(t))$
- **speed:** $v(t) = \|\mathbf{v}(t)\|$
- **accelaration:** $\mathbf{a}(t) = \mathbf{v}'(t)$

Example

The position of the particle is given by $r(t) = (t^3, t^2)$
Find the velocity and acceleration vectors at the point $(8, 4)$.

Geometric point of view

Definition

Curve means a set of $C \subset \mathbb{R}^n$, $n \geq 2$, which can be represented as

$$C = \{\mathbf{r}(t) : t \in I\} = \mathbf{r}(I) = \text{the set of the values of } \mathbf{r},$$

where $I \subset \mathbb{R}$ is an interval, and the function $\mathbf{r}\colon I \to \mathbb{R}^n$ is continuous.

- Here $\mathbf{r} = \mathbf{r}(t)$ is the *parameterization* of the curve C and I is the *parameter interval* corresponding to the parameterization.
- Same curve (as a set) can have different parametrizations
- The parameter interval I can be open (a, b), closed $[a, b]$, or even semi-open $(a, b]$, $[a, b)$. **Most often it is closed**.

About notation

The parameterization of **space curve** (i.e. $n = 3$) can be given as

$$\mathbf{r}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3 \quad \text{when } t \in I,$$

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or it can be given in the so-called coordinate form

$$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \end{cases} \quad \text{when } t \in I,$$

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or the vector form can be used

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad \text{when } t \in I,$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ are the *natural basis vectors* of $\mathbb{R}^3$.

Examples

- Circle in plane

$$textbfv(t) = (x_0 + rcos(t), y_0 + rsin(t)) \in \mathbb{R}^2,$$

where $t \in [0, 2\pi]$, $r > 0$ is the radius of the circle and (x_0, y_0) is the center of the circle.

- Helix curve (helical spring)

$$\mathbf{r}(t) = (a\cos(t), a\sin(t), bt) \in \mathbb{R}^3,$$

where $a, b > 0$ are parameters: a is the radius of the spring and b can be thought of as the elongation of the spring.

- The graph of the real valued one variable function

$$\mathbf{r}(t) = (t, f(t)) \in \mathbb{R}^2$$

Tangent of the curve 1/3

- We study a space curve with a continuously derivable parametrization $\mathbf{r}(t)$
- The secant of the curve corresponding to the parameter interval $[t, t + \Delta t]$ is the vector

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t).$$

As $\Delta t \to 0$, then $\Delta \mathbf{r}$ turns more and more tangent to the curve, but
...
at the same time its length decreases towards zero.

Tangent of curve 2/3

The problem is solved by dividing the expression by Δt:

$$\frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \quad = \left(\frac{x(t + \Delta t) - x(t)}{\Delta t}, \frac{y(t + \Delta t) - y(t)}{\Delta t}, \frac{z(t + \Delta t}{\Delta} \right.$$

Now if we set $\Delta t \to 0$, we get:

$$\lim_{t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = (x'(t), y'(t), z'(t))$$

Curve tangent 3/3

Definition

If the curve $C \subset \mathbb{R}^3$ is continuously derivable parametrization $\mathbf{r}$, then at the point $\mathbf{r}(t)$,

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$$

is the tangent vector of the curve and the functions x, y, z are the coordinate functions of the parametrization.

- The z coordinate is omitted in the case of the plane.
- Compare this to the kinematic point of view
- What happens to the tangent vector if the parametrization of the curve is changed?

Example - Cycloid

https://en.wikipedia.org/wiki/Cycloid

The parameterization of the cycloid (using angle t) is of the form

$$\begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t). \end{cases}$$

Calculate the tangent vector.

Arclength of a curve

Let $\mathbf{r}\colon [a, b] \to \mathbb{R}^n$ be a continuously derivable **one to one** parametrization of the curve C.

Approximate the curve by line segments $\Delta\mathbf{r}_i$

The length of the curve is approximately

$$\sum_i \|\Delta\mathbf{r}_i\| = \sum_i \|\frac{\Delta\mathbf{r}(t_i)}{\Delta t_i}\|\Delta t_i$$

Let the approximation compress. Replacing the sum by an integral we obtain a formula for the length of the curve:

$$\ell(C) = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

Remarks

- Actually parametrization could have finitely many points were one to one property fails. For example think the sign ∞ as a curve. Reason: a single point does not have a length.
- If the parameterization of the curve is only piecewise continuously derivable, we get the total curve length by summing the curve lengths of the continuosly derivable parts.
- Although there are always infinitely many different one-to-one continuously derivable parametrizations of a curve, it can be shown that the length of the curve does not depend on the length of the curve of such a parametrization.

Example - circle

- Circle with radius R can be parametrized by $\mathbf{r}(t) = (R\cos(t), R\sin(t))$ where $t \in [0, 2\pi]$. Calculate its perimeter.
- Make an another paramterization for the same circle and calculate the perimeter using this new parametrization.

Example - graph of a function

For the length of the graph of a continuously derivable function $y = f(x)$ between $[a, b]$ there is a formula already known from high school:

$$\ell = \int_a^b \sqrt{1 + f'(t)^2}\, dt.$$

Where does this come from?

- The graph of the function could be parameterized $\mathbf{r}(t) = (t, f(t))$.
- Now calculate the length of the graph between $[a, b]$ using what we have learned.

Back to kinemetic point of view

The distance travelled by a particle from the time a to time b can be obtained by the formula:

$$\text{distance travelled} = \int_a^b \|\mathbf{r}'(t)\|\, dt$$