Lecture 2: Multivariable functions

Learning goals:

1. What are multivariable functions?
2. How to illustrate multivariable function?
3. What are level curves (contour lines)?
4. How the limits are defined for multivariable function?
5. What kind of techniques there are to calculate multivariable limits?
6. How the continuity is defined for multivariable function?

Where to find the material?

Corral 2.1

Guichard et friends 14.1, 14.2

Active Calculus 9.1, 10.1

Adams-Essex 13.1, 13.2

Multivariable functions

$$f : D \to \mathbb{R} \text{ where } D \subset \mathbb{R}^n (n \geq 2)$$

D is called the domain of f and usually chosen to be the largest set where f could be defined

Examples

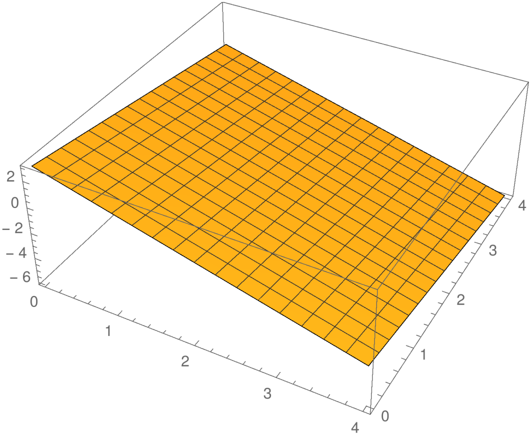
1. The formula $f(r, h) = \pi r^2 h$ describes the function of two variables r, h. Its value is the volume of the cylinder, where r is its radius and h is its height.
2. $f(x_1, x_2, x_3, x_4) = x_1 + x_2 + x_3 + \sqrt{x_4}$

What are the appropriate domains for these functions?

How to illustrate multivariable functions?

- By setting $y = f(x)$ we can make a graph for one variable function (Calculus 1)
- Due to the fact that we live in 3D-world, we can only make graphs for functions with two variables.
- The graph of a two variable function $z = f(x, y)$
- Next some examples of these

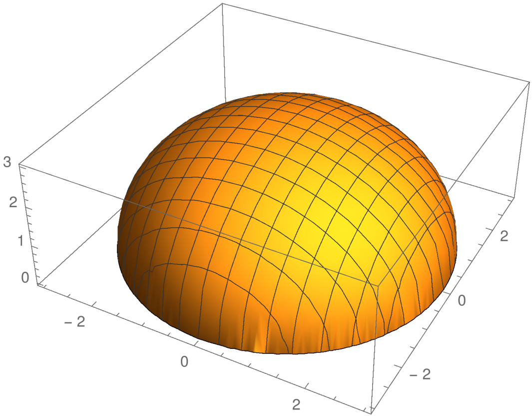
Example 1



The graph $z = f(x, y)$, when

$$f(x, y) = 3\left(1 - \frac{x}{4} - \frac{y}{4}\right).$$

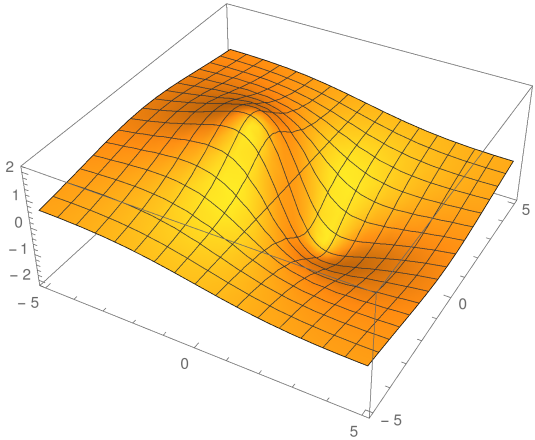
Example 2



The graph $z = f(x, y)$, when

$$f(x, y) = \sqrt{9 - x^2 - y^2}.$$

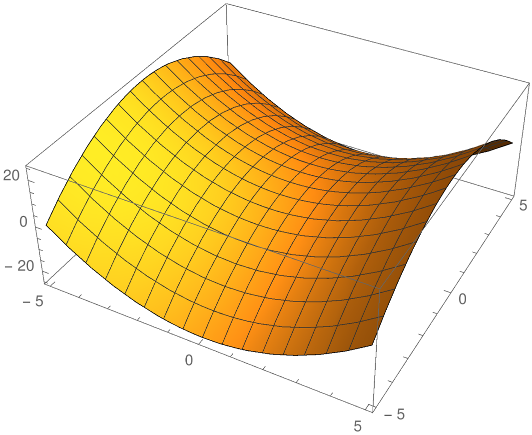
Example 3



The graph $z = f(x, y)$, when

$$f(x, y) = \frac{-6x}{2 + x^2 + y^2}.$$

Example 4

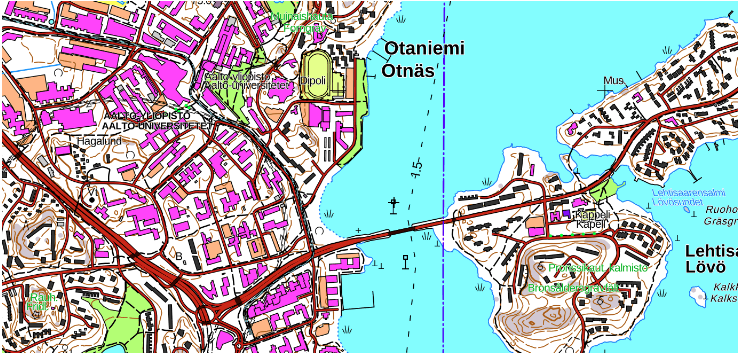


The graph $z = f(x, y)$, when

$$f(x, y) = x^2 - y^2.$$

Level curves (or contour lines)

- Consider $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$
- $c \in \mathbb{R}$ is a constant
- The set $C = \{(x, y) : f(x, y) = c\}$ is often a curve (geometric point of view).
- **For example** The contour lines on a map are the level curves of the altitude of a point on the map (x, y) to the surface of the sea at that point.



Examples

Let's look examples from Active calculus multivariable -book, Chapter 9.1

How to illustrate multivariable functions? (revisited)

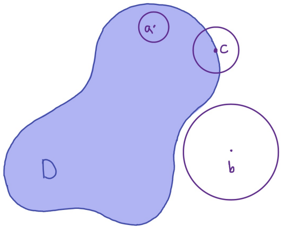
- We can draw the graph of a two variable function $z = f(x, y)$
- Or we can use level curves to illustrate this two variable function
- higher dimension illustrations are harder (level curves are replaced with level sets)

Towards the limits

Before we define the limit let's discuss different type of points for a set D:

- **interior point a**
- **exterior point b**
- **boundary point c**
- **accumulation point**

A point $\mathbf{d}$ is an accumulation point for the set D if for all $r > 0$ $D \cap \{\mathbf{x} \in \mathbb{R}^n : 0 < \|\mathbf{x} - \mathbf{d}\| < r\}$ is not empty.



Limits for multivariable functions

- Let $D \subset \mathbb{R}^n$, $n \geq 2$ and $f\colon D \to \mathbb{R}$ be a function.
- Suppose additionally that the point $\mathbf{x}_0 \in \mathbb{R}^n$ is the **accumulation point** of the set D.

We say that a function f has a limit value L at the point $\mathbf{x}_0$ and denote

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L, \quad \text{where } \mathbf{x} \in D,$$

if for all $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon)$ such that $|f(\mathbf{x}) - L| < \varepsilon$ whenever $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in D$.

Calculation rules for limits

Let $D \subset \mathbb{R}^n$, $n \geq 2$, $\mathbf{x}_0$ be the accumulation point of the set D and $f, g\colon D \to \mathbb{R}$ be such functions, that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = M$. Then:

1. $$\lim_{\mathbf{x} to \mathbf{x}_0} \big(f(\mathbf{x}) \pm g(\mathbf{x})\big) = L \pm M.$$

2. $$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})g(\mathbf{x}) = LM.$$

3. $$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{L}{M}, \text{if } M \neq 0.$$

4. If $L \in (a, b)$ and $F\colon (a, b) \to \mathbb{R}$ is continuous at the point L, then
$$\lim_{\mathbf{x}\to\mathbf{x}_0} F\big(f(\mathbf{x})\big) = F(L).$$

Examples about using calculation rules

- Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 2x$ and $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = y^2$
  What is
  $$\lim_{(x,y)\to(2,3)} (f(x,y) - g(x,y)) = ?$$

- Let $f(x,y) = \frac{x}{y}$ (What is the domain of f?)
  and $F : \mathbb{R} \to \mathbb{R}$, $F(t) = \sin(t)$
  What is
  $$\lim_{(x,y)\to(\pi/3,2)} F(f(x,y)) = ?$$

Techniques for calculating limits

- Examples above where quite easy, but how do we approach questions like:

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = ?$$

- If we do not know what the limit might be, then we could first simplify the situation by approaching $(0,0)$ by a curve and thus reducing the situation to the one variable limit case (that we know how to handle)

Examining limits using curves

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} =?$$

- Simplest curves are x- and y-axis. So let's first approach along them:
- If the point $(0,0)$ is approached from the direction of the x-axis, i.e. along the curve $\mathbf{r}_1(t) = (t,0)$, we get

$$\lim_{t\to 0} f(t,0) = \lim_{t\to 0} \frac{2t\cdot 0}{t^2+0^2} = 0.$$

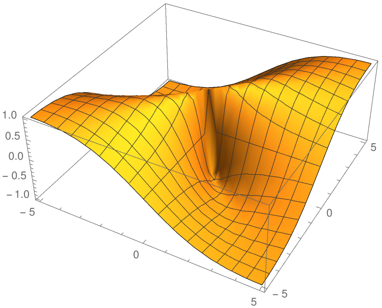
- Along the y-axis, i.e. along the curve $\mathbf{r}_2(t) = (0,t)$, we get

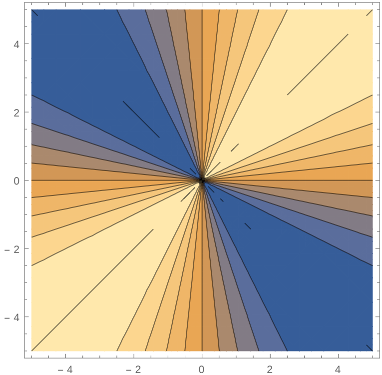
$$\lim_{t\to 0} f(0,t) = \lim_{t\to 0} \frac{2\cdot 0\cdot t}{0^2+t^2} = 0.$$

- On the other hand, the line $x = y$, which can be parametrized by $\mathbf{r}_3(t) = (t,t)$, gives

$$\lim_{t\to 0^+} f(t,t) = \lim_{t\to 0^+} \frac{2t\cdot t}{t^2+t^2} = \lim_{t\to 0^+} \frac{2t^2}{2t^2} = 1.$$

Picture of the situation





The function $f(x, y)$

$$f(x, y) = \frac{2xy}{x^2 + y^2}.$$

does not have limit at $(0, 0)$.

Another example

- Examine the limit of the function

$$f(x,y) = \frac{2x^2y}{x^4+y^2}$$

at the point $(0,0)$.

- The value of the function is 0 on the coordinate axes and thus if we approach along coordinate axes then we get 0.
- Moreovar, for all lines $\mathbf{r}(t) = (t, kt)$ we get

$$f(t,kt) = \frac{2kt^3}{t^4+k^2t^2} = \frac{2kt}{t^2+k^2} \to 0, \text{ when } t \to 0.$$

- But if we approach $(0,0)$ along the curve $\mathbf{r}_1(t) = (t, t^2)$ we get

$$\lim_{t\to 0} f(t,t^2) = \lim_{t\to 0} \frac{2t^4}{t^4+t^4} = 1.$$

- Thus, this function has no limit at the origin.

Techniques for calculating multivariable limits

- In the case of a single variable, one technique to show that there limit does not exists is to compare the right and left limit values.
- This will not work for multiple variables:
  - In general, there are infinitely many directions from which the point $\mathbf{x}_0$ can be "approximated" in the set $D \subset \mathbb{R}^n$, $n \geq 2$
  - The example (titled Another example) shows that even approaching by all possible straight lines is not enough.
- Tools showing that limit exists and is a certain value:
  - Sandwiching with an easier algebraic expression
  - Sometimes polar coordinates might simplify the situation
  - **NOTE:** l'Hôspital does not have multivariable version

Example

Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $f(x,y) = \frac{x^2 y}{x^2+y^2}$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exists? And if so what is the value of it?

Continuity

Definition

Let $D \subset \mathbb{R}^n$, $n \geq 2$ and $\mathbf{x}_0 \in D$. The function $f : D \to \mathbb{R}$ is **continuous at the point $\mathbf{x}_0$** if

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

A function is continuous on the set D if it is continuous at every point in D.

Examples

1. Function
$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$
is continuous at the origin because $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ (calculated earlier)
2. Function
$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} when (x,y) \neq (0,0) \\ 0 \quad \text{when } (x,y) = (0,0) \end{cases}$$
is not continuous at the origin because it does not have limit at the origin (calculated earlier)