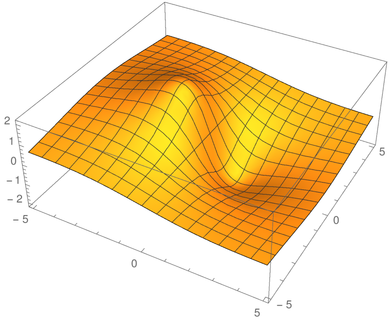
Introduction

$f : \mathbb{R}^n \to \mathbb{R}$

- derivative?
  - partial derivative
  - directional derivative
  - total derivative
- Why needed?
  - optimization: max/min
  - linear approximation



Lecture 3: Partial derivatives and differentiability

Learning goals:

1. What are partial derivatives?
2. What is tangent plane and normal for a graph?
3. How to find the tangent plane and the normal by partial derivatives?
4. What are higher order partial derivatives?
5. How is differentiability defined in multivariable calculus?
6. What is the Jacobian matrix?

Where to find the material?

Corral 2.2, 2.3

Guichard et friends 14.3, 14.6 (does not contain tangent planes)

Active Calculus 10.2, 10.3 and 10.4

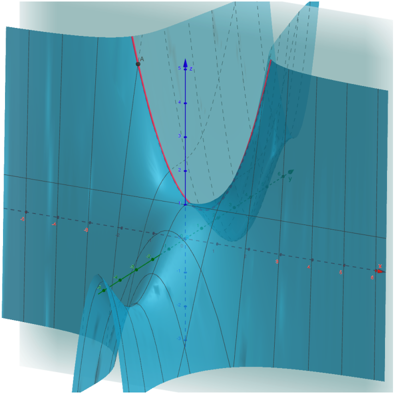
Adams-Essex 13.3, 13.4, partially 13.6

Partial derivatives

- are the simplest derivatives that we have in multivariable calculus
- they mimic the one variable case by keeping all except one variable fixed

Partial derivatives, intro example

- Let's consider
  $f(x,y) = x^2y + \cos(x)$
- idea of partial derivative was keep all except one variable fixed



If we keep y fixed, let's say it is 1, then we have just a normal single variable function $x \mapsto x^2 \cdot 1 + \cos(x)$, whose derivative we can calculate.

Partial derivatives

Let $D \subset \mathbb{R}^n$, $n \geq 2$ and $f\colon D \to \mathbb{R}$ be a function.

Partial Derivative

For all $j = 1, \ldots, n$, the partial derivative of the function f at the point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D$ with respect to the variable x_j is

$$\lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_j, \ldots, x_n)}{h}$$
$$= \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h}$$

if this limit exists.

Notations for partial derivatives

- The partial derivative of the function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ at point x with respect to the variable x_j is denoted usually by

$$\frac{\partial}{\partial x_j} f(x_1, \ldots, x_n) = \partial_{x_j} f(x_1, \ldots, x_n)$$

$$= \partial_j f(x_1, \ldots, x_n) = D_j f(x_1, \ldots, x_n) = f_{x_j}(x_1, \ldots, x_n).$$

- In the case $n = 2$, we often write $z = f(x, y)$, which allows us to use the notation $z = f(x, y)$.

$$\partial_x f(x, y) = \partial_1 f(x, y) = \frac{\partial z}{\partial x}, \quad \partial_y f(x, y) = \partial_2 f(x, y) = \frac{\partial z}{\partial y}.$$

Example 1

Let the function $f\colon \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = x^2 \sin y.$$

What are the partial derivatives of this (at the point (x,y))?

Its partial derivatives are:

$$\partial_x f(x,y) = 2x \sin y$$

and

$$\partial_y f(x,y) = x^2 \cos y.$$

Example 2

What are

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y},$$

when $z = x^3y^2 + x^4y + y^4$?

We get

$$\frac{\partial z}{\partial x} = 3x^2y^2 + 4x^3y$$

and

$$\frac{\partial z}{\partial y} = 2x^3y + x^4 + 4y^3.$$

Example 3

What is $\partial_1 f(0, \pi)$ when $f(x, y) = e^{xy} \cos(x + y)$?

Let's first calculate

$$\partial_1 f(x, y) = ye^{xy} \cos(x + y) - e^{xy} \sin(x + y).$$

Thus

$$\partial_1 f(0, \pi) = \pi e^0 \cos(\pi) - e^0 \sin(\pi) = -\pi.$$

Sometimes the value of the partial derivative at the point $\mathbf{x}_0$ is denoted by

$$\left(\frac{\partial f}{\partial x_j}\right)\bigg|_{\mathbf{x}_0}$$

In above example we could use notation

$$\partial_1 f(0, \pi) = \left(\frac{\partial f}{\partial x_1}\right)\bigg|_{(0,\pi)}$$

Tangents for a surface

- In a single variable case, the derivative can be used to find the tangent to the graph of the function at a given point.
- Let $D \subset \mathbb{R}^2$, $f \colon D \to \mathbb{R}$ and $(a, b) \in D$.
- For a surface $z = f(x, y)$, we get two tangent vectors at the point (a, b):
    - Consider first the curve $\mathbf{r}_1(t) = (t, b, f(t, b))$ i.e. we are letting only x-coordinate to move.
    - Derivative of this curve is $\mathbf{r}_1'(t) = (1, 0, \partial_1 f(t, b))$ and
    - its value when $t = a$ gives the first tangent vector:

$$\mathbf{T}_1 = \mathbf{i} + \partial_1 f(a, b)\mathbf{k}$$

    - Similarly, for the other consider the curve $\mathbf{r}_2(t) = (a, t, f(a, t))$ and obtain

$$\mathbf{T}_2 = \mathbf{j} + \partial_2 f(a, b)\mathbf{k}.$$

Normal for a surface

A normal vector $\mathbf{n} = \mathbf{n}(a, b)$ is perpendicular to both tangent vectors. Therefore, it is obtained as the cross product:

$$\mathbf{n} = \mathbf{T}_1 \times \mathbf{T}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \partial_1 f(a, b) \\ 0 & 1 & \partial_2 f(a, b) \end{vmatrix}$$

$$= -\partial_1 f(a, b)\mathbf{i} - \partial_2 f(a, b)\mathbf{j} + \mathbf{k}.$$

We could have also calculated $\mathbf{T}_2 \times \mathbf{T}_1$ and obtained a normal vector pointing in the opposite direction.

Tangent Plane

- There is two usual ways to get the equation for the tangent plane: 1) using the tangent vectors or 2) using a normal vector
- Let's calculate the tangent plane for a surface $z = f(x, y)$ at point (a, b) using both ways.
- For this denote by $P = (a, b, f(a, b))$.

Tangent plane using tangent vectors

The tangent plane for a surface $z = f(x, y)$ at point (a, b):

- A general point (x, y, z) of a tangent plane has to be form $P + \lambda_1 \mathbf{T}_1 + \lambda_2 \mathbf{T}_2$ for some λ_1 and $_2$
- Solving λ_1 and $_2$ from equation

$$(x, y, z) = (a, b, f(a, b)) + \lambda_1 \mathbf{T}_1 + \lambda_2 \mathbf{T}_2$$

  we obtain $z = f(a, b) + \partial_1 f(a, b)(x - a) + \partial_2 f(a, b)(y - b)$ which is the equation for a tangent plane.

Tangent plane using a normal vector

The tangent plane for a surface $z = f(x, y)$ at point (a, b):

- The tangent plane is always perpendicular to a normal vector.
- Thus a vector from the point P to a general point (x, y, z) of the tangent plane is perpendicular to the normal vector
- So

$$((x, y, z) - P) \cdot \mathbf{n} = 0$$

- Solving z from this yields

$$z = f(a, b) + \partial_1 f(a, b)(x - a) + \partial_2 f(a, b)(y - b)$$

i.e. the equation for the tangent plane.

Example of calculating the normal vector and tangent plane

- Find the tangent plane and a normal for a surface $z = \sin(xy)$ at the point $(\pi/3, -1)$.
- Let's first compute the partial derivatives:

$$\frac{\partial z}{\partial x} = y\cos(xy) \text{ and } \frac{\partial z}{\partial y} = x\cos(xy).$$

- At the point $(\pi/3, -1)$ we get

$$\left.\frac{\partial z}{\partial x}\right|_{(\pi/3,-1)} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(\pi/3,-1)} = \frac{\pi}{6}.$$

- Thus, on the surface in question, at the point $(\pi/3, -1)$, there is a normal vector

$$\mathbf{n} = -(1/2)\mathbf{i} + (\pi/6)\mathbf{j} - \mathbf{k}.$$

- The tangent plane at the point $(\pi/3, -1)$ is

$$z = \frac{-\sqrt{3}}{2} - \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\pi}{6}(y + 1).$$

Higher order partial derivatives

For the function $f : \mathbb{R}^n \to \mathbb{R}$, we can define higher order partial derivatives.

If $z = f(x, y)$, then, for example

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = f_{xx}(x, y)$$

and

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = f_{yx}(x, y).$$

Similarly, if $w = f(x, y, z)$, we get, for example

$$\frac{\partial^5 w}{\partial y \partial x \partial y^2 \partial z} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial f}{\partial z}(x, y, z).$$

Example of higher order partial derivatives

Compute all the second order partial derivatives of the function $f(x, y) = x^3y^4$.

- First order partial derivatives

$$f_x(x, y) = 3x^2y^4 \qquad f_y(x, y) = 4x^3y^3.$$

- Thus

$$f_{xx}(x, y) = \frac{\partial}{\partial x}3x^2y^4 = 6xy^4, \qquad f_{yx}(x, y) = \frac{\partial}{\partial x}4x^3y^3 = 12x^2y^3,$$

$$f_{xy}(x, y) = \frac{\partial}{\partial y}3x^2y^4 = 12x^2y^3, \qquad f_{yy}(x, y) = \frac{\partial}{\partial y}4x^3y^3 = 12x^3y^2.$$

In above example

$$f_{xy}(x, y) = f_{yx}(x, y)$$

This is true in general for nice functions:

Schwarz's Theorem

When the second order partial derivatives are **continuous**, the order of the derivation can be changed.

A proof of this can be found, for example, in a Youtube video here.

Summary of basic tools so far (before going into "total derivative")

1. $f : \mathbb{R} \to \mathbb{R}$
   - is a real-valued function of one variable
   - limits, continuity, derivative (Calculus 1)
2. $f : \mathbb{R} \to \mathbb{R}^n$
   - single-variable vector-valued function
   - by looking coordinate functions returns to the case 1
3. $f : \mathbb{R}^n \to \mathbb{R}$
   - a real-valued function of several variables
   - limits and continuity (previous lecture)
   - partial derivatives ("only one direction at a time", so back to the case 1)
   - "total derivative"
4. $f : \mathbb{R}^n \to \mathbb{R}^m$
   - a vector-valued function of several variables
   - by looking coordinate functions returns to the case 3 for limits, continuity and partial derivatives
   - "total derivative"

Review: definition of a derivative in a one-variable situation

$f: \mathbb{R} \to \mathbb{R}$ f is differentiable at the point $x_0 \in \mathbb{R}$ (i.e. it has a derivative at that point)

1. if the limit value

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =: f'(x_0)$$

   exists

   OR EQUIVALENTLY

2. if there exists $a \in \mathbb{R}$ and a function $\phi: \mathbb{R} \to \mathbb{R}$ such that
   - $f(x_0 + h) = f(x_0) + ah + \phi(h)$ for all $h \in \mathbb{R}$ and
   - $\lim_{h \to 0} \frac{\phi(h)}{h} = 0$

The latter: linear approximation

Differentiable multivariable function

Definition

$f\colon \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at the point $\mathbf{x}_0 \in \mathbb{R}^n$ if there exits a linear mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and a function $\phi\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

- $f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + T(\mathbf{h}) + \phi(\mathbf{h})$ on all $\mathbf{h} \in \mathbb{R}^n$ and
- $\lim_{\mathbf{h}\to\mathbf{0}} \frac{\phi(\mathbf{h})}{||\mathbf{h}||} = \mathbf{0}$

In other words, the function can be approximated near the point $\mathbf{x}_0$ by an affine linear mapping

Matrix presentation for a "total derivative"

- Matrix Algebra: A linear map can be presented by a matrix
- If the function f is differentiable at the point $\mathbf{x}_0$, then a matrix presentation of the linear mapping in the definition is obtained by partial derivatives:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{x}_0) \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}_0) & \frac{\partial f_m}{\partial x_2}(\mathbf{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}_0) \end{bmatrix}$$

This matrix is called the **Jacobian matrix** of the function and is denoted by $Df(\mathbf{x}_0)$.