Lecture 8: Solving a group of equations by Newton's method, double integrals in rectangular domains

Learning goals:

1. What is a multivariable version of Newton's method?
2. How one uses Newton's method to solve system of equations numerically?
3. How double integrals are calculated in rectangular domains?

Where to find the material?

Newton's method: Adams-Essex 14.7 and 14.8

Double integals:

Corral 3.1

Guichard et friends 15.1

Active Calculus 11.1, 11.2

Adams-Essex 15.1, 15.2

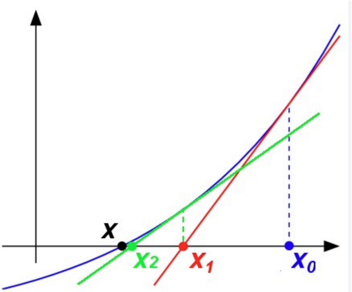
Applications of Partial Derivatives

Newton's method

Newton's method

The Newton's method can be used to find the zeros of the function $f : \mathbb{R} \to \mathbb{R}$, i.e. the solutions of the equation $f(x) = 0$.

- Make initial guess x_0 for the solution of the equation.
- Next we use linear approximation for the function f at this point, that is, the function $l(x) = f(x_0) + f'(x_0)(x - x_0)$.
- Then we look the zero of the linear approximation i.e. solve the equation $l(x_1) = 0$.
- Repeat the previous steps using x_1 instead of x_0 as the initial value, etc.



- This procedure leads to an algorithm where the iteration steps are given by
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots .$$
- The convergence of this sequence and thus locating the zero of the function depends on the initial guess x_0.
- More info: Wikipedia: Newton's method

Example using of a single-variable Newton method

Find the approximate value of $\sqrt{5}$.

- This is then finding a zero for the function $f(x) = x^2 - 5$
- Since $2 = \sqrt{4}$ (quite close to $\sqrt{5}$), thus choose $x_0 = 2$ as a initial guess.
- Since $f(x) = x^2 - 5$, so $f'(x) = 2x$. We obtain the iteration formula

$$x_1 = x_0 - \frac{f(2)}{f'(2)} = 2 - \frac{4-5}{2 \cdot 2} = \frac{9}{4}.$$

$$x_2 = x_1 - \frac{f(9/4)}{f'(9/4)} = \frac{9}{4} - \frac{81/16 - 5}{2 \cdot 9/4} = \frac{161}{72} \approx 2.2361.$$

- Note that $\sqrt{5} \approx 2.236068$, so just two iterations gave a pretty good approximation.

Newton's method multivariable case

Newton's method also works for the function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$, since method is based on the linear approximation. In this case, the derivative in the iteration formula must be replaced by the Jacobian matrix

$$D\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

The iteration formula:

$$\mathbf{x}_{n+1} = \mathbf{x}_n - D\mathbf{f}(\mathbf{x}_n)^{-1}\mathbf{f}(\mathbf{x}_n), \quad n = 0, 1, 2, \dots,$$

where $D\mathbf{f}(\mathbf{x}_n)^{-1}$ is the inverse matrix of $D\mathbf{f}(\mathbf{x}_n)$.

How Newton's method is used to numerical to solve systems of equations

Example

Find a solution for the following system of equations

$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ x^2 + y^2 - z = 1 \\ x + y + z = 3 \end{cases}$$

This system is equivalent to

$$\begin{cases} x^2 + y^2 + z^2 - 3 = 0 \\ x^2 + y^2 - z - 1 = 0 \\ x + y + z - 3 = 0 \end{cases}$$

Thus solving the equation system is equal for finding the zero for

$$\mathbf{f}(x, y, z) = (x^2 + y^2 + z^2 - 3, x^2 + y^2 - z - 1, x + y + z - 3).$$

Let's use the initial guess $\mathbf{x}_0 = (1, 0, 1)$.

- The Jacobian of f is

$$D\mathbf{f}(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- One can calculate

$$\mathbf{x}_1 = (3/2, 1/2, 1), \quad \mathbf{x}_2 = (5/4, 3/4, 1) \text{ and } \mathbf{x}_2 = (9/8, 7/8, 1)$$

  what is the healthiest thing to do on a computer e.g. using MATLAB
- See that the iterations converge towards the point $(1, 1, 1)$, which is the exact (and by all accounts the only) solution to the problem.

Iteraton in example using Python

```
import numpy as np

#Set initial guess
x = np.array([1, 0, 1])

#Loop for calculating the iterations.Example calculates 10 iterations.
for i in range (10):

  # First, calculate the value of function at the point x
  # Note! Problems x, y and z are x=x[0], y=x[1], z=x[2]
  fx = np.array([x[0]**2+x[1]**2+x[2]**2-3, x[0]**2+x[1]**2-x[2]-1, x[0]+x[1]+x[2]-3 ])

  # Jacobian matrix for f at point x
  D = np.array([[2*x[0], 2*x[1], 2*x[2]],[2*x[0], 2*x[1], -1], [1, 1, 1] ])

  # Inverse of the Jacobian matrix
  Dinv = np.linalg.inv(D)

  # Use iteration formula
  x = x - Dinv.dot(fx)

  print(x)
```

Another example of solving a system of equations

Problem: Find a solution, to four decimal places, of the following system of equations

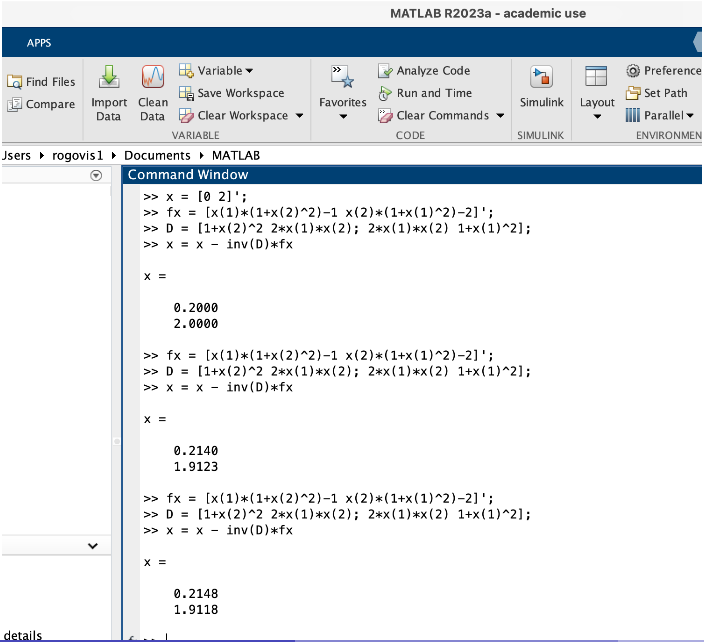
$$\begin{cases} x(1+y^2) & = 1 \\ y(1+x^2) & = 2 \end{cases}$$

- First interpret this as a quest to find a zero point for a function
- $$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad f(x, y) = \big(x(1+y^2) - 1, y(1+x^2) - 2\big)$$
- Do Newton's iteration calculations using MATLAB or Python, for example.

Solution with Python

```
import numpy as np

#Set initial guess
x = np.array([3, 2])

#Loop for calculating the iterations.Example calculates 20 iterations.
for i in range (20):

  # First, calculate the value of function at the point x
  # Note! Problems x and y  are x=x[0], y=x[1]
  fx = np.array([x[0]*(1+x[1]**2)-1, x[1]*(1+x[0]**2)-2 ])

  # Jacobian matrix for f at point x
  D = np.array([[1+x[1]**2, 2*x[0]*x[1]],[2*x[0]*x[1], 1+x[0]**2] ])

  # Inverse of the Jacobian matrix
  Dinv = np.linalg.inv(D)

  # Use iteration formula
  x = x - Dinv.dot(fx)

  print(x)
```

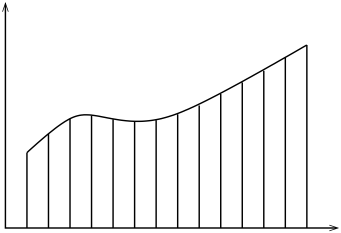
Solution with Matlab



Integral calculus

First recall: the case of a single variable

- In the case of a single variable, the integral is given by *Riemann sums* as a limit value



- Formally

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

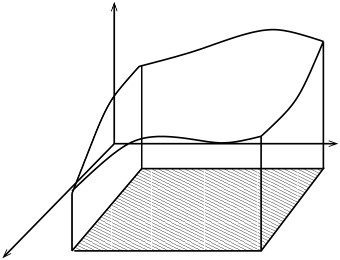
where $a = x_0, x_1, \ldots, x_n = b$ is the even division of the interval $[a, b]$ and Δx is the length of the division interval.

Double integral

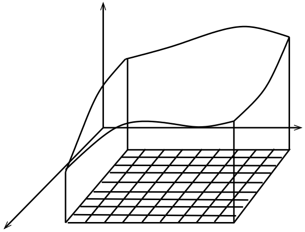
- Let $D \subset \mathbb{R}^2$ be a set in the plane and $f \colon D \to \mathbb{R}$. We want to define *the double integral*

$$\iint_D f(x,y)\,dA.$$

- Let us first consider the special case, where f getsonly positive values and $D = [a,b] \times [c,d]$:



- Let's divide the plane subset $D = [a, b] \times [c, d]$ equally into a grid with n vertices on each axis:



- Now we can define (similarly than in single variable case)

$$\iint_D f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\, \delta x \delta y,$$

where Δx and Δy correspond to the length of the intervals in the x and y directions:

$$\Delta x = \frac{b-a}{n}, \quad \Delta y = \frac{d-c}{n}.$$

Notes

- Physically the integral then gives the volume that is between the graph of the function and D.
- Physically we could also think f is a surface density for a rectangular plate D, then the integral would give the mass of the plate.
- Like in single variable cases, we can extend the definition of the double integral to the case where f gets also negative values.

How to calculate double integrals?

- In the single-variable case, we have the fundamental theorem of Analysis:

$$f(x) = \frac{d}{dx} \int_c^x f(t)\, dt, \text{ when } c, x \in [a, b]$$

  and $f : [a, b] \to \mathbb{R}$ is a continuous function.
- This implies that the integration and derivation are counter-operations of each other, leading to many formulas useful for integration.
- Unfortunately, there is no single, clear-cut equivalent to the fundamental theorem of analysis in the multivariable case.
- BUT ...

Fubini's Theorem

If $f(x,y)$ is a continuous function on a rectangle $D = [a,b] \times [c,d]$, then

$$\iint_D f(x,y) = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

- Thus the double integral can be calculated as an iterated integral.
- In the last two the inner integral is calculated first and then the one whose notation is outer.
- It does not matter in which order we integrate. (Sometimes one order is easier than the other.)

More info

Wikipedia page for Fubini's theorem

Active Calculus Multivariable see section 11.2

Example

Let $f(x, y) = xy^2$. Calculate

$$\iint_D f(x, y)\, dA, \text{ where } D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

First, write the equalization integral as a double integral, and calculate

$$\iint_D xy^2\, dA = \int_0^1 \int_0^1 xy^2\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2 y^2}{2}\right]_{x=0}^1 dy$$

$$= \int_0^1 \frac{y^2}{2}\, dy = \left[\frac{y^3}{6}\right]_{y=0}^1 = \frac{1}{6}.$$

Example of the effect of integration order

Calculate
$$\int \int_S xcos(xy)\, dA,$$
where $S = \{(x, y) \in \mathbb{R}^2 : x \in [0, \pi], y \in [0, 1]\}$.

Hint: Integrating first with respect to y is much easier.