Lecture 9: The triple integral, integration in general regions and improper integrals

Learning goals:

1. How to calculate the triple integrals in a rectangle?
2. How do you calculate integrals in general regions?
3. How are improper integrals calculated for double and triple integrals?

Where to find the material?

Corral 3.2, 3.3

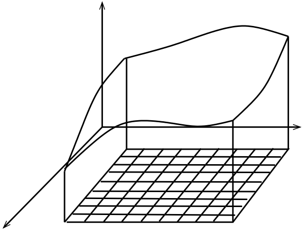
Guichard et friends 15.1, 15.5

Active Calculus 11.3, 11.7

Adams-Essex 15.2, 15.3, 15.5

Last time: double integral in a rectangle

- Divide the plane subset $D = [a, b] \times [c, d]$ equally into a grid such that there are n division points on each axis:



- Now we can define:

$$\iint_D f(x, y)\, dA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta x \Delta y,$$

where Δx and Δy correspond to the length of the intervals in the x and y directions:

$$\Delta x = \frac{b-a}{n}, \quad \Delta y = \frac{d-c}{n}.$$

Calculating the double integrals

- Fubini's theorem:
- When $D = [a, b] \times [c, d]$ and f continuous we have

$$\iint_D f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

- Calculation in on order might be easier.

Triple integral

- In the same way, one can define triple integral:

$$\iiint_D f(x,y,y)\,dV = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta x \Delta y \Delta z,$$

when $D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{R}^3$ and $f\colon D \to \mathbb{R}$. Here

$$\Delta x = \frac{b_1 - a_1}{n}, \quad \Delta y = \frac{b_2 - a_2}{n} \text{ and } \Delta z = \frac{b_3 - a_3}{n}.$$

- This can be generalized to case $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$, where $n \geq 2$.

Calculating the triple integrals

- These can be calculated as iterated integrals similarly as double integrals when our function f is continuous:
- $$\iiint_D f(x,y,z)\,dV = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y,z)\,dx\,dy\,dz,$$

  when $D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$.
- Or any other order. The order mighteffect how easy the integral is to calculate.
- This follows from Fubini's theorem.

Example

Let $f(x,y,z) = xye^z$. Evaluate

$$\iiint_D f(x,y,z)\,dV, \text{ where } D = [0,2]\times[0,1]\times[-1,1].$$

Let's first write the integral as an iterated integraland then calculate:

$$\iiint_D xye^z\,dV = \int_{-1}^{1}\int_0^1\int_0^2 xye^z\,dx\,dy\,dz$$

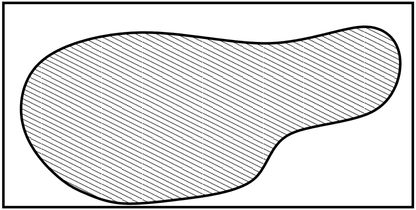
$$= \int_{-1}^{1}\int_0^1 \frac{x^2ye^z}{2}\bigg|_{x=0}^{2} dy\,dz = \int_{-1}^{1}\int_0^1 2ye^z\,dy\,dz$$

$$= \int_{-1}^{1} y^2e^z\bigg|_{y=0}^{1} dz = \int_{-1}^{1} e^z\,dz$$

$$= e^z\bigg|_{z=-1}^{1} = e - e^{-1}.$$

Integration in more general regions - theory

- Consider a function $f\colon D \to \mathbb{R}$ defined on a subset D of a plane (or space).
- So far it has been assumed that D is a rectangle. If D is not a rectangle, we can consider a rectangle $\hat{D}$ that contains D i.e. for which $D \subset \hat{D}$.
- For an integral to be defined, the set D must be "clean" (e.g., it suffices that the edge is piecewise smooth).



- Let us define the function $\hat{f}\colon \hat{D} \to \mathbb{R}$ as follows:

$$\hat{f}(x,y) = \begin{cases} f(x,y), & \text{when} \quad (x,y) \in D, \\ 0, & \text{when} \quad (x,y) \in \hat{D} \setminus D. \end{cases}$$

- Now we can define

$$\iint_D f(x,y)\,dA := \iint_{\hat{D}} \hat{f}(x,y)\,dA.$$

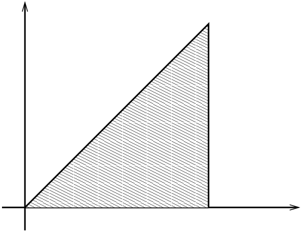
- In the same way, the triple integral can also be defined for the case of a non-rectangular region:

$$\iiint_D f(x,y,z)\,dV := \iiint_{\hat{D}} \hat{f}(x,y,z)\,dV,$$

where $\hat{D}$ is rectangular parallelepiped and $D \subset \hat{D}$.

Example

Let $D = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < x\}$. Evaluate the integral of $f(x, y) = xy$ over D.



$$\iint_D xy\, dA = \int_0^1 \left(\int_0^x xy\, dy \right) dx$$

$$= \int_0^1 \frac{xy^2}{2} \bigg|_{y=0}^{x} dx = \int_0^1 \frac{x^3}{2}\, dx = \frac{x^4}{8} \bigg|_{x=0}^{1} = \frac{1}{8}.$$

Integration can be done in other order too:

$$\iint_D xy\,dA = \int_0^1 \left(\int_y^1 xy\,dx \right) dy$$

$$= \int_0^1 \frac{x^2 y}{2} \bigg|_{x=y}^{1} dy = \int_0^1 \frac{y}{2} - \frac{y^3}{2}\,dy$$

$$= \left[\frac{y^2}{4} - \frac{y^4}{8} \right]_{y=0}^{1} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

Another example

Evaluate the integral

$$I = \int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy\, dx$$

We cannot do the integration in this order, so we express I as a double integral

$$\iint_D e^{y^3}\, dA$$

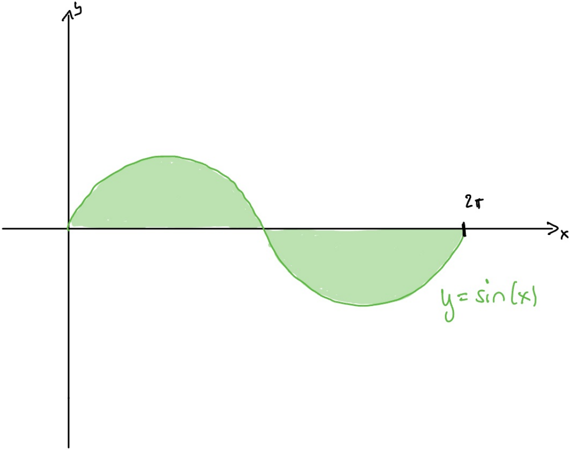
and identify D (draw a picture).

Change the order of integration.

Answer $\frac{e-1}{3}$

Third example

What is a volume of solid, whose height is 1 and whose bottom is in a picture:

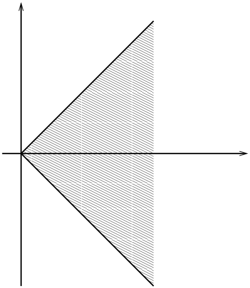


Improper integrals

- To simplify matters, we have assumed so far that our domain is bounded and also the function is bounded (and continuous)
- As in the single variable case, improper multivariable integrals can arise if
  - either the domain of integration is unbounded
  - or the integrand is unbounded near any point of the domain or its boundary
- An improper integral of a function f satisfying $f \geq 0$ on a domain D must either exists (i.e. to converge to a finite value) or be infinite

Improper integrals, unbounded domain

- Consider only the case where the function f is non-negative, i.e. $f(\mathbf{u}) \geq 0$ for all $\mathbf{u} \in D$.
- Calculate the integral of the function $f(x, y) = e^{-x^2}$ in the domain in the region bounded by the lines $y = \pm x$ in the unbounded domain D, where $x > 0$.



If the integral convergences, its value is obtained by calculating

$$\iint_D e^{-x^2}\, dA = \int_0^\infty \int_{-x}^{x} e^{-x^2}\, dy\, dx = \int_0^\infty 2x e^{-x^2}\, dx$$

$$= \lim_{R\to\infty} \int_0^R 2xe^{-x^2}\, dx.$$

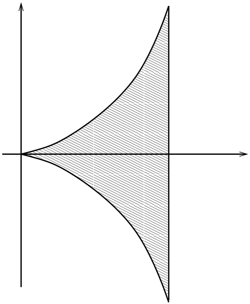
Notice that $\frac{d}{dx} e^{-x^2} = -2xe^{-x^2}$.

Thus

$$\lim_{R\to\infty} \int_0^R 2xe^{-x^2}\, dx = \lim_{R\to\infty} -e^{-x^2}\Big|_{x=0}^{R} = \lim_{R\to\infty} 1 - e^{-R^2} = 1.$$

Improper integrals, unbounded functions

Let $D_1 = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ |y| \leq x^2\}$ and unbounded function $f(x, y) = 1/x^2$.



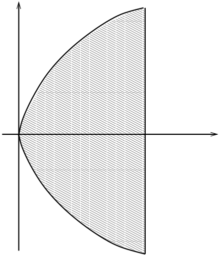
Evaluate

$$\iint_{D_1} \frac{1}{x^2}\, dA = \int_0^1 \int_{-x^2}^{x^2} \frac{1}{x^2}\, dy\, dx$$

$$= \int_0^1 2\, dx = 2.$$

2nd example, unbounded function

Evaluate the integral of $f(x) = \frac{1}{x^2}$ in a domain $D_2 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, |y| \leq \sqrt{x}\}$.



$$\iint_{D_2} \frac{1}{x^2}\, dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{x^2}\, dy\, dx$$

$$= \int_0^1 2\sqrt{x}\frac{1}{x^2}\, dx = \int_0^1 2x^{-3/2}\, dx = \lim_{\varepsilon \to 0+} \frac{2x^{-1/2}}{-1/2}\Bigg|_{x=\varepsilon}^{1} = \infty.$$

The convergence of the integral depends on the integrand and also domain.