# Lecture 10: Change of variables in integrals in general

Learning goals:

1. Why the change of variables might be useful?
2. How the change of variables is done?

**Where to find the material?**

Corral 3.5

Guichard et friends 15.7

Active Calculus 11.9

Adams-Essex 15.4, 15.6

# Why is a change of variable needed?

- can make the calculation of the integral easier
- In a single-variable case: integration can be made easier by a procedure called substitution
- In multivariable integrals, the change of variables also changes the shape of the integration set $\rightarrow$, an attempt can be made to obtain a rectangle as the integration set

# Change of variables in one variable case

- If $f$ is continuous and $g$ continuously differentiable on a closed interval $[a, b]$, then

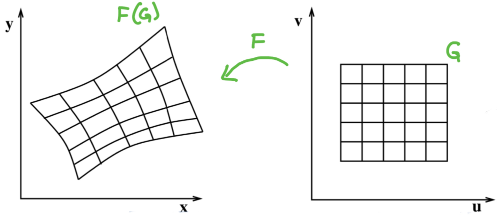
$$\int_A^B f(x)\,dx = \int_a^b f(g(u))g'(u)\,du,$$

  where $A = g(a)$ ja $B = g(b)$.
- also called **the method of subsitution**
- is obtained from the chain rule
- **Example:** Evaluate using the method of subtitution

$$\int_0^2 \frac{x}{x^2+1}\,dx.$$

# Change of variables in double integrals

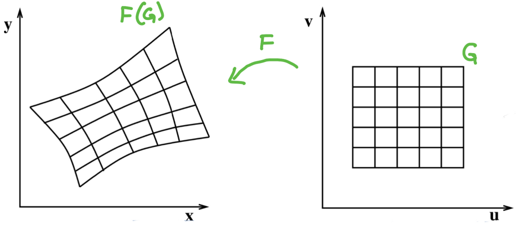
$$\iint_D f(x,y)\,dx\,dy = \iint_G f(\mathbf{F}(u,v))???\,du\,dv, \text{ where } D = \mathbf{F}(G)$$




What kind of the change of variable function **F** can be?

# Change of variables function

- Function $\mathbf{F}\colon G \to D \subset \mathbb{R}^2$, where $G \subset \mathbb{R}^2$
- Let's assume that all first order partial derivatives of $\mathbf{F}$ are continuous
- And that $\mathbf{F}$ is bijection (= one-to-one and onto) i.e. for every point $(x, y) \in D$ there is exactly one point in $(u, v) \in G$ such that $\mathbf{F}(u, v) = (x, y)$.



# Change of variables in double integral

$$\iint_{\mathbf{F}(G)} f(x,y)\,dx\,dy = \iint_G f(\mathbf{F}(u,v))???\,du\,dv.$$

What should be put to replace ??? ?

For this we need information what **F** does to the surface area.

Let's draw a picture and calculate.

# Jacobian determinant

- Earlier on this course, we had Jacobian matrix:
- $f : \mathbb{R}^n \to \mathbb{R}^n$

$$Df(\mathbf{x}_0) = \begin{bmatrix} \partial_1 f_1(\mathbf{x}_0) & \partial_2 f_1(\mathbf{x}_0) & \cdots & \partial_n f_1(\mathbf{x}_0) \\ \partial_1 f_2(\mathbf{x}_0) & \partial_2 f_2(\mathbf{x}_0) & \cdots & \partial_n f_2(\mathbf{x}_0) \\ \vdots & & \ddots & \vdots \\ \partial_1 f_m(\mathbf{x}_0) & \partial_2 f_m(\mathbf{x}_0) & \cdots & \partial_n f_m(\mathbf{x}_0) \end{bmatrix}$$

- The determinant of this matrix is called the **Jacobian determinant**.
- The absolute value of the Jacobian determinant $|\det D\mathbf{F}(u, v)|$ tells us *local* surface area change

# Change of variable formula for double integrals

**Change of variable formula**

$$\iint_D f(x,y)\,dx\,dy = \iint_G g(u,v)\big|\det D\mathbf{F}(u,v)\big|\,du\,dv$$

where $g(u,v) = f(x(u,v), y(u,v)) = f(\mathbf{F}(u,v))$ ja $D = \mathbf{F}(G)$.
And $\mathbf{F}(u,v) = (x(u,v), y(u,v))$ is bijection between $G$ and $D$ with continuous 1st order partial derivatives.

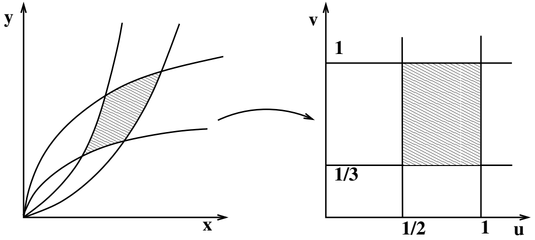
- It is not necessary for $G$ and $D$ be closed or that transformation is one-to-one on the boundary
- How to use this formula? In practice, it is often easier to first think the inverse of $\mathbf{F}$. Let's look an example.

# Example

Find the area of the finite plane region $D$ bounded by the four parabolas $y = x^2$, $y = 2x^2$, $x = y^2$ and $x = 3y^2$.
We get the area from the integral $\iint_D 1\, dA$
Using mapping

$$\mathbf{G}(x, y) = u\mathbf{i} + v\mathbf{j}, \quad u(x, y) = \frac{x^2}{y},\ v(x, y) = \frac{y^2}{x}.$$

we notice that the region corresponds to the rectangle
$G = [1/2, 1] \times [1/3, 1]$




Over the rectangle it is much easier to integrate.

For the change of variable formula, we want the inverse $\mathbf{F} = \mathbf{G}^{-1} \colon G \to D$

The matrix algebra gives

$$\det D\mathbf{F}(u, v) = \frac{1}{\det D\mathbf{G}(x, y)}.$$

Calculate

$$\frac{\partial u}{\partial x} = \frac{2x}{y}, \quad \frac{\partial u}{\partial y} = -\frac{x^2}{y^2}.$$

We also get

$$\frac{\partial v}{\partial x} = -\frac{y^2}{x^2}, \quad \frac{\partial v}{\partial y} = \frac{2y}{x}.$$

Thus

$$\det D\mathbf{G}(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix}$$

$$= 4 - 1 = 3, \text{ so } \left|\det D\mathbf{F}(u,v)\right| = \frac{1}{3}.$$

Finally

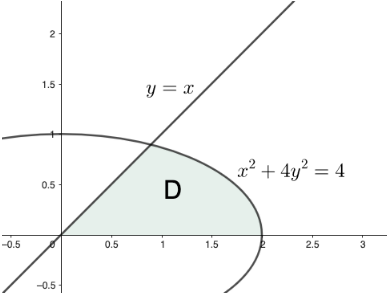
$$\iint_D 1\, dA = \iint_G \frac{1}{3}\, du\, dv = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{9}.$$

# Another example

Calculate

$$\iint_D \frac{y}{x}\, dA$$

where $D$ is the green region in the picture




Answer: $\frac{1}{4}\ln 5$

# Third example

Evaluate the integral

$$\iint_R e^{\frac{x-y}{x+y}}\, dA,$$

where $R = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y \leq 1\}$.

Answer: $\frac{e^2-1}{4e}$ (This example is from Corral: Vector calculus Example 3.9)

# Change of variables in triple integral

## Change of variables formula

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_G g(u,v,w)\Big|\det D\mathbf{F}(u,v,w)\Big|\,du\,dv\,dw$$

where $g(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w) = f(\mathbf{F}(u,v,w))$ and $D = \mathbf{F}(G)$.
And $\mathbf{F}$ is bijection between $G$ and $D$ with continuous 1st order partial derivatives.

The most commonly used changes are switching to polar coordinates in the case of the double integral and to cylindrical or spherical coordinates in the case of the triple integral.

# Notation of the Jacobian determinant

- In some books (and some STACK problems) use the following notation for the Jacobian determinant of the change of the variable function
$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du\, dv := \left|\det D\mathbf{F}(u,v)\right| du\, dv$$
- So the change of variable formula looks like
$$\iint_D f(x,y)\, dx\, dy = \iint_G g(u,v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, du\, dv$$
- The idea behind the notation is that the $x$ and $y$ are changed to $u$ and $v$.
- There is also the three variable version of this notation.