Exam 1
Intermediate Macroeconomics I, 2023

Answer all questions and make sure your handwriting is legible. Maximum points are the same for all problems.

1) Consider the standard Solow growth model. The law-of-motion for capital is given by:

$$K_{t+1} = sF(A_t, K_t, L_t) + (1-\delta)K_t$$

where $0 < \delta < 1$ is the depreciation rate of capital, K_t, $0 < s < 1$ is the savings rate in the economy. A_t is the level of technology which evolves exogenously according to $A_{t+1}/A_t = 1 + a$, L_t is the size of the labor force which evolves according $L_{t+1}/L_t = 1 + n$. Consumption, C_t, is given by $Y_t - S_t = (1-s)F(A_t, K_t, L_t)$. Finally, production is given by $Y_t = F(A_t, K_t, L_t) = K_t^\alpha (A_t L_t)^{1-\alpha}$.

Determine the steady state value of $K/(AL)$, and the growth rate of output per labor $(\frac{Y_t/L_t - Y_{t-1}/L_{t-1}}{Y_{t-1}/L_{t-1}})$. Explain why technological progress is necessary for explaining increasing output-per-labor in steady state.

Solution: The steady state $K/(AL) = k^*$ is given by

$$k^* = \left(\frac{s}{\delta + a + n + an}\right)^{\frac{1}{1-\alpha}} \approx \left(\frac{s}{\delta + a + n}\right)^{\frac{1}{1-\alpha}}$$

which can be obtained from the law-of-motion for capital by multipling the left-hand-side by $\frac{(1+a)(1+n)}{A_t L_t (1+a)(1+n)}$ and right-hand-side by $\frac{1}{A_t L_t}$ to define the law-of-motion in terms of capital per efficient labor unit, $k_t = K_t/(A_t L_t)$, then setting $k_t = k_{t+1} = k^*$ and solving for k^* (note: it is ok to use approximations in this question).

The steady state growth rate of output per labor is

$$\frac{Y_t/L_t - Y_{t-1}/L_{t-1}}{Y_{t-1}/L_{t-1}} = \frac{A_t(k^*)^\alpha - A_{t-1}(k^*)^\alpha}{A_{t-1}(k^*)^\alpha} = a$$

since $Y_t/L_t = A_t(k^*)^\alpha$.

The solution above reveals why technological progress ($a > 0$) is essential for explaining increasing output-per-labor in steady state. If $a = 0$ then growth in output-per-labor would be zero.

2) Consider a representative firm which has the AK production function: $Y_t = AK_t L$, where $A > 0$, L is constant and K_t is capital. Further suppose that the firm operates under perfect competition in goods, labor and capital markets and only produces if profits are non-negative (i.e. $AKL - wL - rK \geq 0$). Can we "microfound" the AK model under this assumption that the firm operates under perfect competition? Demonstrate by solving the firm profit-maximization problem.

Solution: Under perfect competition, the first-order conditions are

$$\begin{aligned} AK &= w \\ AL &= r \end{aligned}$$

Substituting r and w into the profit function gives

$$AKL - wL - rK = AKL - AKL - ALK = -AKL \leq 0$$

Clearly profits are negative if production is positive. Therefore, we cannot microfound the AK model by assuming perfect competition. Imperfect competition, externalities, or some alternative assumptions are needed to support this model.

Note that we could instead assume that the firm chooses K only, then profits will equal $-wL < 0$, and so the result is the same in this case.

3) Consider a representative household that tries to optimally choose consumption in period 1, c_1, and period 2, c_2, and savings in period 1, a, in order to maximize utility

$$U(c_1, c_2) = log(c_1) + \beta log(c_2)$$

subject to the budget constraints

$$\begin{aligned} c_1 + a &= y_1 \\ c_2 &= (1 + r(1 - \tau))a + y_2 \end{aligned}$$

where $0 < \tau < 1$ is a tax on the savings rate, r, and $\beta(1+r) = 1$ is assumed for convenience.

Solve for optimal consumption, c_1 and c_2, when $\tau = 0$ and when $\tau > 0$. How does the result change if $\tau > 0$ versus if $\tau = 0$? Explain in terms of income and substitution effects.

Solution: The solution can be obtained in a variety of ways. Most approaches first involve combining the two period's budget constraints. For example, we can substitute $a = y_1 - c_1$ into the period 2 budget constraint which yields

$$c_2 = (1 + r(1 - \tau))(y_1 - c_1) + y_2$$

Substituting the last expression for c_2 into the utility function gives $U = log(c_1) + \beta log((1 + r(1 - \tau))(y_1 - c_1) + y_2)$. Taking the first order condition with respect to c_1 then gives:

$$\begin{aligned} &\frac{1}{c_1} - \frac{\beta(1 + r(1 - \tau))}{c_2} = 0 \\ &\rightarrow c_2 = \beta(1 + r(1 - \tau))c_1 \end{aligned}$$

Note that this collapses to $c_1 = c_2$ if $\tau = 0$. Substituting the last expression for c_2 into the budget constraint $c_2 = (1 + r(1 - \tau))(y_1 - c_1) + y_2$ and solving for c_1 yields

$$c_1 = \frac{1}{1 + \beta}\left(y_1 + \frac{y_2}{1 + r(1 - \tau)}\right)$$

Substituting this solution for c_1 into the first-order condition yields the solution for c_2

$$c_2 = \frac{\beta}{1+\beta}\left(y_1(1+r(1-\tau))+y_2\right)$$

Setting $\tau = 0$ gives the solution when there is no tax. It is apparent from the two solutions that raising τ leads to lower period 2 consumption and higher period 1 consumption. The tax is akin to an interest rate decrease and therefore it induces a substitution effect that makes period 1 consumption cheap relative to period 2 consumption (c_1 rises and c_2 falls). The effectively lower interest rate has a positive income effect for borrowers and negative income effect for savers.

4) Extend the OLG model by assuming that households live for three periods. Households work when they are "young" and "middle-age", then retire during the last period of their life. Young and middle-age households pay pension contributions and old households receive pension benefits, which are a fraction, $\bar{\epsilon}$, of the wage they earned when they were middle-aged and employed. The wage rate earned by *all* employed households at time t is $w_t = w_{t-1}(1+a)$ and the population of young households at t is $N_t = N_{t-1}(1+n)$.

Assume a PAYG pension system. Write down the budget constraint for this system (pension contributions on left-hand-side, and pension benefits on right-hand-side). What is the pension contribution rate, τ, that young and middle-age households must pay as a fraction of their wage to fund the system? How would this fraction change if *only* the middle-aged households need to pay?

Solution: The budget constraint is

$$\tau\left(w_t N_t + w_t N_{t-1}\right) = \bar{e} w_{t-1} N_{t-2}$$

which implies

$$\tau = \frac{\bar{e}}{(1+a)(1+n+(1+n)^2)}$$

Alternatively, if middle-aged households provided all pension contributions then the budget constraint becomes

$$\tau w_t N_{t-1} = \bar{e} w_{t-1} N_{t-2}$$

which implies

$$\tau = \frac{\bar{e}}{(1+a)(1+n)}$$

5) Suppose money demand, M^d, is given by the following equation: $M_t^d = \kappa P_t Y_t$ where $\kappa > 0$, $Y_t = Y_{t-1}(1+g)$ given $g \geq 0$ is real GDP, and P_t is the price level at time t. Suppose money market equilibrium is satisfied in each period, t: $M_t^s = M_t^d$ where M_t^s is the supply of money at t.

a) Suppose M^s is fixed at $\bar{M}$. What is the rate of inflation, $\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}$? Explain whether a permanent one-time increase in M^s causes the *long-run* inflation rate to increase.

Solution: if $M_t^s = \bar{M} = \kappa P_t Y_t = \kappa P_{t-1} Y_{t-1}$ then

$$\frac{\kappa P_t Y_t}{\kappa P_{t-1} Y_{t-1}} = (1+g)(1+\pi) = 1$$

Therefore, $\pi = \frac{1}{1+g} - 1 = \frac{-g}{1+g} \approx -g$ (note: it is ok to use approximations in this question).

A permanent one-time increase in the inflation rate does not increase long-run inflation. This because after the increase occurs, we have $M_t = M_{t-1}$ in all periods and the same result from before holds. However, increasing the money supply will raise inflation in the period the increases occurs.

b) Suppose now that government targets a constant rate of inflation, $\pi^* > 0$, using a money supply rule of the form: $\frac{M_t}{M_{t-1}} = 1 + m$. What value of m is needed to achieve this rate of inflation according to the model?

Solution: From the quantity theory:

$$(1+g)(1+\pi^*) = \frac{\kappa P_t Y_t}{\kappa P_{t-1} Y_{t-1}} = \frac{M_t}{M_{t-1}} = 1 + m$$

Hence, we need: $m = (1+g)(1+\pi^*) - 1 \approx g + \pi^*$