Lecture 12: Applications of integrals and summary of the course

Learning goals:

1. What are some of the practical applications of integral calculus?
2. Overview of the course

Where to find the material?

Corral 3.5

Guichard et friends 15.2, 15.6

Active Calculus 11.5, 11.8

Adams-Essex 15.4, 15.6

Applications that we have seen so far

- The area of a plane region D:

$$\text{Area}(D) = \iint_D 1\, dA.$$

- The volume of a three-dimensional object:

$$\text{Volume}(D) = \iiint_D 1\, dV.$$

- The mass of a body K:

$$m(K) =_K \delta(x, y, z)\, dV,$$

  where $\delta(x, y, z)$ is the density at the point (x, y, z).

- Next, we will discuss some other applications of the integraion that may come up in physics courses, for example.

Other applications

- center of mass
- centroid
- moment of inertia

Center of mass

- Center of mass describes the average location of the mass of the body.
- If the body is supported from the center of mass it stays balanced.
- https://en.wikipedia.org/wiki/Center_of_mass
- The centre of mass $(\bar{x}, \bar{y}, \bar{z})$ of a three-dimensional object D can be calculated from

$$\bar{x} = \left[\iiint_D x\delta(x,y,z)\,dV \right] \left[\iiint_D \delta(x,y,z)\,dV \right]^{-1},$$

$$\bar{y} = \left[\iiint_D y\delta(x,y,z)\,dV \right] \left[\iiint_D \delta(x,y,z)\,dV \right]^{-1},$$

$$\bar{z} = \left[\iiint_D z\delta(x,y,z)\,dV \right] \left[\iiint_D \delta(x,y,z)\,dV \right]^{-1},$$

  where $\delta = \delta(x,y,z)$ is the density of the body at the (x,y,z).
- Similarly can be calculated the center of mass for a lamina (a thin plate)
- For more info, see the Chapter 3.6 in Corral.

Example

- Calculate the center of mass for a unit cube given by the inequalities $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$ when the density is $\delta(x, y, z) = z$.
- First calculate the total mass of the cube

$$\iiint_D \delta \, dV = \int_0^1 \int_0^1 \int_0^1 z \, dx \, dy \, dz$$

$$= \int_0^1 z \, dz = \Big|_{z=0}^{1} \frac{1}{2} z^2 = \frac{1}{2}.$$

- Thus

$$\bar{x} = \frac{\int_0^1 \int_0^1 \int_0^1 xz \, dx \, dy \, dz}{1/2} = \frac{\left(\Big|_{x=0}^{1} \frac{1}{2} x^2 \right) \left(\Big|_{z=0}^{1} \frac{1}{2} z^2 \right)}{1/2}$$

$$= \frac{(1/2)^2}{1/2} = \frac{1}{2}.$$

- Like wise

$$\bar{y} = \frac{\int_0^1 \int_0^1 \int_0^1 yz \, dx \, dy \, dz}{1/2} = \frac{(1/2)^2}{1/2} = \frac{1}{2}.$$

- And

$$\bar{z} = \frac{\int_0^1 \int_0^1 \int_0^1 z^2 \, dx \, dy \, dz}{1/2} = \frac{\Big|_{z=0}^{1} \frac{1}{3} z^3}{1/2}$$

$$= \frac{(1/3)}{(1/2)} = \frac{2}{3}.$$

- So the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = (1/2, 1/2, 2/3).$$

Centroid

- The centroid (also called the geometric center) is the aritmetic mean of all the points of the body.
- https://en.wikipedia.org/wiki/Centroid
- The centroid $(\bar{x}, \bar{y}, \bar{z})$ is calculated with integrals:

$$\bar{x} = \frac{\iiint_D x\, dV}{V}, \qquad \bar{y} = \frac{\iiint_D y\, dV}{V} \text{ and}$$

$$\bar{z} = \frac{\iiint_D z\, dV}{V},$$

  where V is the volume of the body.
- If the body has uniform densitity (i.e. the density $\delta(x, y, z) = \mathrm{constant}$) then the centroid and the center of mass is the same.
- For a plane region the formulas are:

$$\bar{x} = \frac{\iint_D x\, dA}{A} \quad \text{and} \quad \bar{y} = \frac{\iint_D y\, dA}{A},$$

  where A is the area of D.

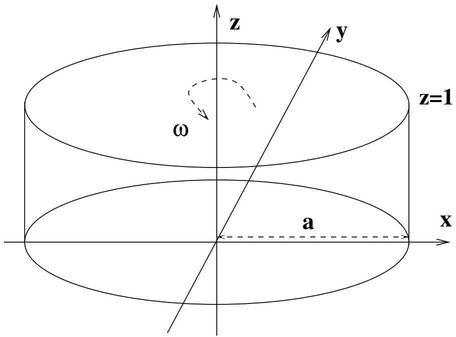
Moment of inertia

- https://en.wikipedia.org/wiki/Moment_of_inertia (See the subsection: Motion in a fixed plane)
- Moment of inertia of a body D when it is rotating around z-axis:

$$I_z(D) = \iiint_D \delta(x, y, z)(x^2 + y^2)\, dV,$$

where $\delta(x, y, z)$ is the density at the point (z, y, z).

Example



- Calculate the moment of the inertia for the cylinder

$$D = \{(x, y, z) : x^2 + y^2 \leq a^2 \text{ ja } 0 \leq z \leq 1\}, \qquad a > 0$$

when rotating around z-axis and when the density is constant i.e. $\delta = \delta_0$.

Solution

- Calculate using the cylinder coordinates

$$
\begin{aligned}
I_z &= \iiint_D (x^2 + y^2)\delta_0 \, dV = \delta_0 \int_0^1 \int_0^{2\pi} \int_0^a r^2 r \, dr \, d\theta \, dz \\
&= 2\pi\delta_0 \int_0^a r^3 \, dr = \frac{1}{2}\pi\delta_0 a^4.
\end{aligned}
$$

- To see that this is really a familiar formula, let's write it into the other form.
- The mass of the cylinder is

$$
m = \iiint_D \delta_0 \, dV = \delta_0 \mathrm{Volume}(D) = \delta_0 \pi a^2.
$$

- Thus we can write

$$
I_z = \frac{1}{2}(\pi\delta_0 a^2)a^2 = \frac{1}{2}ma^2.
$$

Study tips for the exam

To study for the exam

1. Make sure that you know by heart all the concepts (example what is a Jacobian matrix). The following summary have them listed.
2. Make sure that you know the answers to How-questions in the following summary.
3. Look through the exercises. Make again with out looking the model the ones that you did not know how to do in the first. Look the model solution after you have seriously tried to solve the problem yourself.
4. For the extra training, you can look Old exams from Tenttiarkisto or do the problems in the Open Access books.

Summary of the essentials of the course

Curves

- Kinematic point of view: a continuous one variable vector valued function gives a path of a particle, example $\mathbf{r}(t) = (t^2, t-3)$
  - How to calculate speed of a particle? A velocity? An acceleration?
  - How to calculate the distance travelled by the particle?
- Geometric point of view: A curve is a set that can be parametrized with a continuous one variable vector valued function, for example a circle of radius 2 can be parametrized by $r(t) = (2\cos(t), 2\sin(t))$, where $t \in [0, 2\pi]$
  - How to calculate the tangent of the curve?
  - How to calculate the length of the curve?

Multivariable functions

- How to calculate the limit of a multivariable function especially with two variables?
    - What strategies there are to show that the limit exits?
    - What startegies there are to show that the limit doe not exists?
- How to see if a multivariable function is continuous?

Differential calculus

- Concepts: partial derivatives, directed derivative, gradient, Jacobian matrix, Hessian matrix, Jacobian determinant
- How to calculate these?
- How to use the chain rule of differentiation in the multivariable case?

Applications of differential calculus

- Concepts: linear approximation, Taylor polynomials, Newton's method, optimization(critical points, extreme values, second derivative test, Lagrange multipliers)
- How to use linear approximation to get approximation or to estimate error?
- How to calculate Taylor polynomials for multivaraible functions?
- What is the idea of Newton's method for finding the numerical solutions for an equation/a system of equations?
- Optimization:
    - What where the places where the extremes of a function can occur?
    - How to find extremes of the function when
        - the domain of function is whole space?
        - the domain of a function is a set with boundary?
        - there is a constraint for variables?
    - How to use the second derivative test?
    - How to use the Lagrange multiplier -method?

Integral calculus

- How to calculate double integrals?
- How to calculate triple integrals?
- How to do the change of variables in the integration?
  - Polar coordinate change
  - Cylinder coordinates change
  - Spherical coordinate change
- What is the physical meaning of the integrals (especially when the integrand is positive)?
- Applications: How to calculate area, volume or the mass of certain region or solid?