Section 2.7 (from I. Stewart - Algebraic Number Theory)

Exercises covered : 1, 3, 6

Recommended : (all of them but in particular)
4, 5, 8, 10, 11

Hints

1. Which of the following complex numbers are algebraic? Which are algebraic *integers*?
   (a) $355/113$
   (b) $e^{2\pi i/23}$
   (c) $e^{\pi i/23}$
   (d) $\sqrt{17}+\sqrt{19}$
   (e) $(1+\sqrt{17})/(2\sqrt{-19})$
   (f) $\sqrt{(1+\sqrt{2})}+\sqrt{(1-\sqrt{2})}$.

a) —

b) Primitive root of unity

c) Theorem 2.10

d) —

e) Consider for instance the number field
$K = \mathbb{Q}(\sqrt{17}, \sqrt{-19})$ and $\alpha = (1+\sqrt{17})/(2\sqrt{-19}) \in K$.
$N_K(\alpha) \in \mathbb{Z}$?

f) —

3. Find all monomorphisms $\mathbf{Q}(\sqrt[3]{7}) \to \mathbf{C}$.

Theorem 2.4.

4. Find the discriminant of $\mathbf{Q}(\sqrt{3}, \sqrt{5})$.

Recall $\{1, \frac{1+\sqrt{5}}{2}\}$ is an integral basis for $\mathbb{Q}(\sqrt{5})$;
maybe use the fact that $k = \mathbb{Q}(\sqrt{5})(\sqrt{3})$

5. Let $K = \mathbf{Q}(\sqrt[4]{2})$. Find all monomorphisms $\sigma : K \to \mathbf{C}$ and the minimum polynomials (over $\mathbf{Q}$) and field polynomials (over K) of (i) $\sqrt[4]{2}$ (ii) $\sqrt{2}$ (iii) 2 (iv) $\sqrt{2}+1$. Compare with Theorem 2.6.

For the monomorphisms: Theorem 2.4.

8. Compute integral bases and discriminants of

(a) $\mathbf{Q}(\sqrt{2}, \sqrt{3})$

(b) $\mathbf{Q}(\sqrt{2}, i)$

(c) $\mathbf{Q}(\sqrt[3]{2})$

(d) $\mathbf{Q}(\sqrt[4]{2})$.

a) b) Similar to 4

c) Similar to example 2.22.

d) Use 5

10. If $\alpha_1, \ldots, \alpha_n$ are $\mathbf{Q}$-linearly independent algebraic integers in $\mathbf{Q}(\theta)$, and if

$$\Delta[\alpha_1, \ldots, \alpha_n] = d$$

where d is the discriminant of $\mathbf{Q}(\theta)$, show that $\{\alpha_i, \ldots, \alpha_n\}$ is an integral basis for $\mathbf{Q}(\theta)$.

Lemma 1.15