**Optimization Final Exam**

April 20th, 2023

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*The exam is 3 hours and has a total of 120 points. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.*

1. Warm-up

    (a) *(10 points)* Describe, as best you can, how Bellman equations and the principle of optimality are used in optimization. A few sentences is sufficient.

    (b) *(10 points)* Set $X \subseteq \mathbb{R}^n$ is convex. What does that mean?

    (c) *(10 points)* Write down the KKT conditions for

    $$\max x^3 y \text{ s.t. } x^2/2 + y^2/3 \leq 1$$

2. At the University of Maryland, all students are given $T$ "terp bucks". These can be spent on one of two goods, meals at the dining hall (good x) or goods at the university convenience store (good y). In addition, money can be spent at the university convenience store and the cafeteria. A student has utility function $u(x, y)$, strictly increasing, strictly concave, twice continuously differentiable, and in addition to their $T$ terp bucks has budget $M$. Good $x$ and $y$ both have price 1 in dollars, but, in order to encourage students to spend terp bucks on meals, good $x$ has price 1 in terp bucks while good $y$ has price $p > 1$ in terp bucks. So the

consumer solves

$$\max_{x_1,x_2,y_1,y_2} u(x_1+x_2, y_1+y_2)$$
$$\text{s.t. } x_1 + y_1 \leq M$$
$$x_2 + py_2 \leq T$$
$$x_1, x_2, y_1, y_2 \geq 0$$

(a) *(10 points)* Show that this has a unique solution for any $p > 1$.

(b) *(10 points)* Write down the KKT conditions for this problem.

(c) *(25 points)* Using the KKT conditions, show that if $x_1$ and $x_2$ are non-zero then $y_2$ must be 0.

3. There is a single firm selling to a unit mass of consumers. Each consumer has a willingness to pay $\theta \in [0,1]$, uniformly distributed. Time is discrete, and infinite, i.e. $t = 0, 1, 2, \ldots$. In each period, the firm sets a price $p_t \in \mathbb{R}_+$ and all consumers with a willingness to pay above $p_t$ buy the product and leave the market. The firm discounts profits at rate $\delta < 1$. In each period, the state is fully described by the highest willingness to pay remaining, $\bar{\theta}_t$.

   To summarize, at time $t$, all remaining consumers have willingness to pay $\theta \in [0, \bar{\theta}_t]$, where $\bar{\theta}_t$ is determined by the firm's past pricing choices. In that period, the firm sets price $p_t$, and firm receives a payoff of $\delta^t p_t(\bar{\theta}_t - p_t)$. All consumers with willingness to pay above $p_t$ buy the product and leave, and $\bar{\theta}_{t+1}$ adjusts accordingly.

   (a) *(5 points)* As a benchmark, suppose the firm is myopic, $\delta = 0$, so the firm only cares about maximizing their current period profits. Describe the optimal price as a function of the highest willingness to pay $\bar{\theta}$.

   (b) *(5 points)* Set up the Bellman equation for this problem.

   (c) *(15 points)* Use the Bellman equation to show that the value function is increasing in $\bar{\theta}$. Argue that at a give $\bar{\theta}$, the optimal price is higher than the price you found in (a) for any $\delta > 0$.

(d) *(20 points)* Using the envelope theorem, find an expression for $\bar{\theta}_{t+2}$ as a function of $\bar{\theta}_t$ and $\bar{\theta}_{t+1}$. This, along with the conditions that $\bar{\theta}_0 = 1$ and $\lim_{t\to\infty} \bar{\theta}_t = 0$ this pins down the optimal price path.