Rate of Convergence (Q-Converence)

Let $\{x^k\}$ be any infinite sequence. Let $s^k = \max_{l \geq k} x^l$ and define

$$\limsup_{k \to \infty} x^k = \lim_{k \to \infty} s^k.$$

Sometimes $\limsup$ is written as $\overline{\lim}$. A $\limsup$ always exists (if we allow $+\infty$). If $\lim x^k = L$ then $\limsup x^k = L$, but the opposite is not true.

Examples: (1) $x^k = (-1)^k$, then $s^k = 1$ and $\limsup x^k = 1$.

(2) $x^k = \sin(k)$ then $s^k = 1$ and $\limsup x^k = 1$ (note in this case there is no obvious convergence).

Assume $\lim_{k \to \infty} x^k = \hat{x}$ and there is some M such that $x^k \neq \hat{x}$ for all $k > M$. Then for $p \geq 0$ let

$$C(p) = \limsup_{k \to \infty} \frac{|x^{k+1} - \hat{x}|}{|x^k - \hat{x}|^p}.$$

Then, if $C(p^*) < \infty$ for some p^* then $C(p) = 0$ for $p < p^*$. If $C(p^*) > 0$ for some p^* then $C(p) = \infty$ for $p > p^*$. Both of these results come from the equality

$$\frac{|x^{k+1} - \hat{x}|}{|x^k - \hat{x}|^p} = \frac{|x^{k+1} - \hat{x}|}{|x^k - \hat{x}|^{p^*}} |x^k - \hat{x}|^{p^* - p}.$$

So, there exists a p^* (possibly infinite) such that

$$C(p) = \begin{cases} 0 & \text{if } 0 \leq p < p^* \\ C(p^*) & \text{if } p = p^* \\ \infty & \text{if } p > p^* \end{cases}.$$

This number p^* is the *order of convergence* for the sequence x^k and determines the rate of convergence as follows:

- If $p^* = 1$ and $C(1) = 1$ then we say the convergence is sublinear.
- If $p^* = 1$ and $1 > C(1) > 0$ then we say the convergence is linear.
- If $p^* > 1$ or $C(1) = 0$ then we say the convergence is superlinear.
- If $p^* = 2$ then we say the convergence in quadratic.
- If $p^* = 3$, convergence is cubic, etc.

When working with convergence estimates it is often useful to use the following approximation:

$$|x^{k+1} - \hat{x}| \approx C|x^k - \hat{x}|^{p^*}$$

for some constant C (not necc. $C(p^*)$).