Optimization Final Exam

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The exam is 3 hours and has a total of 120 points. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Warm-up

   (a) *(10 points)* Describe, as best you can, how Bellman equations and the principle of optimality are used in optimization. A few sentences is sufficient.

   Solution. *The principle of optimality prescribes that we can break a dynamic maximization problem up into a series of simpler static subproblems. This is done through the Bellman equation, which leverages the recursive structure of the dynamic problem to break up the problem into the payoff from initial choices and the value of all subsequent choices (captured through the value function). This establishes a recursive formula for the value function, which in finite horizon problems can be solved by working backwards from the last period and in infinite horizon problems is the unique solution to a functional equation that can be solved using value function iteration.*

   (b) *(10 points)* Set $X \subseteq \mathbb{R}^n$ is convex. What does that mean?

   Solution. *X is convex if for any $x, y \in X$, $\lambda x + (1-\lambda)y \in X$ for any $\lambda \in (0,1)$.*

   (c) *(10 points)* Write down the KKT conditions for

$$\max x^3 y \text{ s.t. } x^2/2 + y^2/3 \leq 1$$

Solution. *The KKT conditions are*

$$\begin{aligned}
3x^2y &= \lambda x \\
x^3 &= 2\lambda y/3 \\
\lambda(1 - x^2/2 - y^2/3) &= 0 \\
\lambda &\geq 0
\end{aligned}$$

2. At the University of Maryland, all students are given T "terp bucks". These can be spent on one of two goods, meals at the dining hall (good x) or goods at the university convenience store (good y). In addition, money can be spent at the university convenience store and the cafeteria. A student has utility function $u(x, y)$, strictly increasing, strictly concave, twice continuously differentiable, and in addition to their T terp bucks has budget M. Good x and y both have price 1 in dollars, but, in order to encourage students to spend terp bucks on meals, good x has price 1 in terp bucks while good y has price $p > 1$ in terp bucks. So the consumer solves

$$\begin{aligned}
&\max_{x_1,x_2,y_1,y_2} u(x_1 + x_2, y_1 + y_2) \\
&\text{s.t. } x_1 + y_1 \leq M \\
&x_2 + py_2 \leq T \\
&x_1, x_2, y_1, y_2 \geq 0
\end{aligned}$$

   (a) *(10 points)* Show that this has a unique solution for any $p > 1$.

   Solution. *Be careful, the objective is not strictly concave. Suppose there were two maximizes (x_1', x_2', y_1', y_2') and $(x_1'', x_2'', y_1'', y_2'')$. It can't be that $x_1' + x_2' \neq x_1'' + x_2''$ or $y_1' + y_2' \neq y_1'' + y_2''$, since then the convex combination of the two maximizers would give a strictly higher value for the objective, by strict concavity and the convexity of the feasible set. The only case that remains is $x_1' + x_2' = x_1'' + x_2'' = x$ and $y_1' + y_2' = y_1'' + y_2'' = y$. Since u is strictly increasing, both budget constraints must hold with equality. Simply adding up the two* *constraints immediately gives $x+y+(p-1)y_2' = x+y+(p-1)y_2''$, so $y_2' = y_2''$, which then implies the other three variables must also be the same.*

(b) *(10 points)* Write down the KKT conditions for this problem.

Solution. *We have 6 constraints, let λ_1 and λ_2 be the multipliers on the budget constraints and $\mu_1, \mu_2, \mu_3, \mu_4$ be the multipliers on the non-negativity constraints (in other order they are listed in the statement of the problem). Let u_x be the derivative of $u(x, y)$ with respect to the first component and u_y be the derivative with respect to the second. The KKT conditions are*

$$
\begin{aligned}
&u_x(x_1+x_2, y_1+y_2) = \lambda_1 - \mu_1 \\
&u_x(x_1+x_2, y_1+y_2) = \lambda_2 - \mu_2 \\
&u_y(x_1+x_2, y_1+y_2) = \lambda_1 - \mu_3 \\
&u_y(x_1+x_2, y_1+y_2) = p\lambda_2 - \mu_4 \\
&\lambda_1(M - x_1 - y_1) = 0 \\
&\lambda_2(T - x_2 - py_2) = 0 \\
&\mu_1 x_1 = 0,\ \mu_2 x_2 = 0,\ \mu_3 y_1 = 0,\ \mu_4 y_2 = 0
\end{aligned}
$$

and positivity of all multipliers.

(c) *(25 points)* Using the KKT conditions, show that if x_1 and x_2 are non-zero then y_2 must be 0.

Solution. *If $x_1, x_2 \neq 0$ then $\mu_1 = \mu_2 = 0$. Therefore, $\lambda_1 = \lambda_2$. Now, suppose that $y_2 > 0$. Then $\mu_4 = 0$. Combining the two y focs this gives*

$$\lambda_1 - \mu_3 = p\lambda_1.$$

So $0 \leq -\mu_3 = (p-1)\lambda_1 > 0$ (where the strict inequality follows from strict concavity of u), which is a contradiction.

3. There is a single firm selling to a unit mass of consumers. Each consumer has a willingness to pay $\theta \in [0, 1]$, uniformly distributed. Time

is discrete, and infinite, i.e. $t = 0, 1, 2, \ldots$. In each period, the firm sets a price $p_t \in \mathbb{R}_+$ and all consumers with a willingness to pay above p_t buy the product and leave the market. The firm discounts profits at rate $\delta < 1$. In each period, the state is fully described by the highest willingness to pay remaining, $\bar{\theta}_t$.

To summarize, at time t, all remaining consumers have willingness to pay $\theta \in [0, \bar{\theta}_t]$, where $\bar{\theta}_t$ is determined by the firm's past pricing choices. In that period, the firm sets price p_t, and firm receives a payoff of $\delta^t p_t(\bar{\theta}_t - p_t)$. All consumers with willingness to pay above p_t buy the product and leave, and $\bar{\theta}_{t+1}$ adjusts accordingly.

(a) *(5 points)* As a benchmark, suppose the firm is myopic, $\delta = 0$, so the firm only cares about maximizing their current period profits. Describe the optimal price as a function of the highest willingness to pay $\bar{\theta}$.

Solution. *If the firm is myopic, they maximize $p(\bar{\theta} - p)$, which has FOC $\bar{\theta} = 2p$ so $p = \bar{\theta}/2$.*

(b) *(5 points)* Set up the Bellman equation for this problem.

Solution. *The Bellman equation for this problem is*

$$V(\bar{\theta}) = \max_{p \in [0, \bar{\theta}]} p(\bar{\theta} - p) + \delta V(p)$$

(c) *(15 points)* Use the Bellman equation to show that the value function is increasing in $\bar{\theta}$. Argue that at a give $\bar{\theta}$, the optimal price is higher than the price you found in (a) for any $\delta > 0$.

Solution. *Consider $\bar{\theta} > \bar{\theta}'$. Let $p(\bar{\theta}')$ be the optimal price at $\bar{\theta}'$. Then the firm receives a payoff of $p(\bar{\theta}' - p) + \delta V(p(\bar{\theta}'))$. Since the firm is optimizing at $\bar{\theta}$, we know that*

$$V(\bar{\theta}) \geq p(\bar{\theta}')(\bar{\theta} - p(\bar{\theta}')) + \delta V(p(\bar{\theta}')) \geq V(\bar{\theta}'),$$

since they could always choose price $p(\bar{\theta}')$, and this inequality is *strict unless $p = 0$ which is clearly never optimal at positive $\bar{\theta}$. Therefore V must be increasing.*

The FOC for this problem is $\bar{\theta} - 2p + \delta V'(p) = 0$. So $p > \bar{\theta}/2$ since $V'(p) > 0$.

(d) *(20 points)* Using the envelope theorem, find an expression for $\bar{\theta}_{t+2}$ as a function of $\bar{\theta}_t$ and $\bar{\theta}_{t+1}$. This, along with the conditions that $\bar{\theta}_0 = 1$ and $\lim_{t\to\infty} \bar{\theta}_t = 0$ this pins down the optimal price path.

Solution. *The envelope theorem gives us that $V'(\bar{\theta}) = p(\bar{\theta})$. Moreover, we know that $p(\bar{\theta}_t) = \bar{\theta}_{t+1}$. Therefore, combining this with the first order condition:*

$$\bar{\theta}_t - 2\bar{\theta}_{t+1} + \delta V'(\bar{\theta}_{t+1}) = 0$$

gives $\bar{\theta}_t - 2\bar{\theta}_{t+1} + \delta\bar{\theta}_{t+2} = 0$ so $\bar{\theta}_{t+2} = \frac{1}{\delta}\left(2\bar{\theta}_{t+1} - \bar{\theta}_t\right)$.