Lecture 1. Information Aggregation in Auction

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1 General Setting

N bidders $\{1, ..., N\}$ and one good

Each bidder i privately observes a signal $s_i \in [0, 1]$.

Vector of signals $S = \{s_1, ..., s_N\}$

- Joint density f,
- Marginal density f_i,
- Assume f is affliated ("positive correlation").
  - For any pair of $i, j \in \{1, ..., N\}$ and $x, x', y, y' \in [0, 1]$ with $x > x'$ and $y > y'$,

$$\frac{f(s_i = x, s_j = y, S_{-\{i,j\}})}{f(s_i = x', s_j = y, S_{-\{i,j\}})} > \frac{f(s_i = x, s_j = y', S_{-\{i,j\}})}{f(s_i = x', s_j = y', S_{-\{i,j\}})}$$

  - A higher s_j makes a higher s_i more likely.

Bidder i receives value $v_i(S) = v_i(s_1, ..., s_N)$ when winning the good.

- v_i is increasing in s_j for all j and strictly increasing in s_i

We have considered **private** value case with **independent** signals,

$$v_i(S) = v_i(s_i),$$

$$f = \prod_i f_i.$$

- It captures the case where bidders have different tastes.

We now consider the **pure common** value case where

$$v_i(S) = v(S).$$

- Bidders have the same preference but receive different aspects of information.
- Mineral rights

We would like to examine whether the auction successfully aggregates dispersed information among bidders,

- Whether the price determined in auction reflects the true value $v(S)$ as $N \to \infty$ (the market is large, perfect competition among bidders).
- Efficient market hypothesis: price should contain all information.

2 A Binary-State Model

Unknown state $\omega \in \{\ell, h\}$ with prior q_0 and $1 - q_0$.

Conditional on the state w, signals are independent and identically distributed with density $g(\cdot|\omega)$ and c.d.f $G(\cdot|\omega)$,

$$f(S|\omega) = \prod_i g(s_i|\omega),$$

$$f(S) = q_0 f(S|\ell) + (1-q_0) f(S|h) = q_0 \prod_i g(s_i|\ell) + (1-q_0) \prod_i g(s_i|h).$$

- Signals are correlated through the realization of the true state.
- Signals satisfy **Monotonic Likelihood Ratio Property** (MLRP):

$$\frac{g(s|h)}{g(s|\ell)} \text{ is strictly increasing in } s.$$

  - Higher signal is a stronger indicator for state h.

Common values: $0 \leq v_\ell < v_h$.

$$\begin{aligned} v(S) &= E(v_\omega|S) \\ &= Pr(\omega = \ell|S) v_\ell + Pr(\omega = h|S) v_h. \end{aligned}$$

3 First Price Auction

Suppose bidders follow a strictly increasing bidding strategy $\beta(\cdot)$.

- Symmetric equilibrium

Denote $Y_{1:N} = \max\{s_1, ..., s_N\}$ as the first order statistic, the winning bid is

$$\beta(Y_{1:N})$$

As N increases, we have a sequence of equilibria $\Gamma = \{\beta_N\}_{N=1}^{\infty}$

The first price auction **aggregates information** if for each $\omega \in \{\ell, h\}$ and all $\epsilon > 0$

$$\lim_{N \to \infty} Pr(|\beta(Y_{1:N}) - v_\omega| < \epsilon | \omega) = 1,$$

- The winning bid converges in probability to v_ω.
- It happens when all signals are public.

Proposition 1. *(Milgrom 1979) For every sequence of equilibria* Γ*, the first price auction aggregates information if and only if the signal is unboundedly informative,*

$$\lim_{x \to 1} \frac{g(x|h)}{g(x|l)} = \infty.$$

Information gest aggregated and the market is efficient if and only if there is a fully-revealing signal ($s = 1$) for state h.

We only prove the necessity. Assume

$$\lim_{x \to 1} \frac{g(x|h)}{g(x|l)} = \mu < \infty.$$

Note that if bidder i with signal x wins the auction, then his expected value is

$$E(v_\omega | s_i = x, Y_{1:N-1} \leq x).$$

- The bidder calculates his expected value conditional on (1) his own signal, and (2) he wins the auction x, that is, all other bidders receive signals less than x.
- Winner's curse: if you win, it means that all the bidders do not receive signals as good as yours.

Note that each bidder must receive a non-negative expected payoff,

$$\beta_N(x) \leq E(v_\omega | s_i = x, Y_{1:N-1} \leq x).$$

We have

$$E(v_\omega | s_i = x, Y_{1:N-1} \leq x) = E(v_\omega | Y_{1:N} = x).$$

Consider the case in which bidder i received the highest signal 1,

$$\beta_N(1) \leq E(v_\omega | s_i = 1, Y_{1:N-1} \leq 1) = E(v_\omega | Y_{1:N} = 1).$$

However,

$$E(v_\omega | Y_{1:N} = 1) = E(v_\omega | s_i = 1) = \frac{q_0 v_\ell + (1 - q_0)\mu v_h}{q_0 + (1 - q_0)\mu} < v_h.$$

Therefore, the price in any sequence of the equilibria cannot converge to v_h, and information aggregation fails.

Good news: Information gest aggregated and the market is efficient under a certain condition.

Bad news: The condition is very strong, requiring a fully-revealing signal ($s = 1$) for state h.

- The winner's curse with a large number of bidders is very strong, we hence need a very strong signal to offset it.
- Note that if we don't have the unboundedly informative signal, even when we choose $x \approx 1$,

$$\lim_{n\to\infty} E(v_\omega | s_i = x, Y_{1:N-1} \leq x) = v_\ell$$

4 Second Price Auction

There exists a unique symmetric equilibrium in which bidder i with $s_i = x$ bids

$$\beta(x) = v(x,x) = E(v_\omega | s_i = x, Y_{1,N-1} = x).$$

- $\beta(x)$ is strictly increasing in x.

To prove the validity of this equilibrium. Consider the simple case where there are two bidders 1 and 2 with $s_i = x$ and $s_j = y$. Assume bidder 2 bids

$$\beta(y) = v(y,y) = E(v_\omega | s_1 = y, s_2 = y)$$

First, consider the situation where $x < y$, bidder 1 loses the auction by bidding any number less than $v(y,y)$ including bidding $v(x,x)$. In this case, he gets 0. To win the auction, bidder 1 must bid more than $v(y,y)$ and pay $v(y,y)$. However, in this case, the expected value of the good is

$$E(v_\omega | s_1 = x, s_2 = y) < E(v_\omega | s_1 = y, s_2 = y) = v(y,y).$$

Bidder 1 receives a negative expected payoff. Hence, it is optimal for bidder 1 to bid $v(x,x)$.

Consider the situation where $x > y$, bidder 1 wins the auction by bidding any number larger than $v(y,y)$ including bidding $v(x,x)$. In this case, he receives a positive expected payoff since

$$E(v_\omega | s_1 = x, s_2 = y) > E(v_\omega | s_1 = y, s_2 = y) = v(y,y).$$

Hence, it is optimal for bidder 1 to bid $v(x,x)$.

For the case where $x = y$. Bidder 1's expected payoff is always 0 no matter what he bids.

For the case with more than 2 bidders, one can replace bidder 2 in the above reasoning by the bidder who has the highest signal among the rest $N-1$ bidders.

We then have a sequence of equilibria $\Gamma = \{\beta_N\}_{N=1}^{\infty}$. However, this sequence of equilibria fails to aggregate information if

$$\lim_{x\to 1}\frac{g(x|h)}{g(x|l)}=\infty.$$

Consider the case where

$$\lim_{x\to 1}\frac{g(x|h)}{g(x|l)}=\mu<\infty.$$

Note that

$$E(v_\omega|s_i=x,Y_{1,N-1}=x)=E(v_\omega|Y_{1,N}=x,Y_{2,N}=x)$$

We then have

$$E(v_\omega|Y_{1,N}=1,Y_{2,N}=1)=E(v_\omega|s_i=1,s_j=1)=\frac{q_0v_\ell+(1-q_0)\mu^2v_h}{q_0+(1-q_0)\mu^2}<v_h$$

Hence,

$$\beta_N(1)<v_h.$$

Information fails to aggregate.

5 Large Auctions

Pesendorfer and Swinkels (1997): the market analyzed before is not large enough.

- Only one good for sale.

Consider a model with N bidders and k identical goods. Each good has the same value v_ω to bidders, which depends on the unknown state $\omega\in\{h,\ell\}$.

Uniform price auction: The k highest bidders each get a good and pay a price equal to the $k+1$ highest bid.

There exists a unique symmetric equilibrium where bidder i with signal $s_i=x$ bids

$$\beta_{N,k}(x)=E(v_\omega|s_i=x,Y_{k:N-1}=x).$$

Note that

$$\beta_{N,k}(x)=E(v_\omega|Y_{k:N}=Y_{k+1:N}=x).$$

Now there are two effects:

- Winner's curse: if I win, there are at least $N-k$ bidders receiving signals smaller than mine, which is bad. Hence, I should lower my bid.
- Losser's curse: if I lose, there are at least k bidders receiving signals larger than mine, which is good. Hence, I should increase my bid.

The loser's curse cancels out the winner's curse and hence contributes to information aggregation.

We have,

$$\beta_{N,k}(x) \leq E(v_\omega|Y_{k:N} \geq x, Y_{k+1:N} = x) = E(v_\omega|Y_{k+1:N} = x),$$

$$\beta_{N,k}(x) \geq E(v_\omega|Y_{k:N} = x, Y_{k+1:N} \leq x) = E(v_\omega|Y_{k:N} = x).$$

We therefore have a "sandwich" condition.

$$E(v_\omega|Y_{k:N} = x) \leq \beta_{N,k}(x) \leq E(v_\omega|Y_{k+1:N} = x).$$

Consider a sequence of trades $\{N, k(N)\}_{N=1}^{\infty}$. Under the uniform price auction, there exists a sequence of symmetric equilibria $\Gamma = \{\beta\}_{N=1}^{\infty}$ where

$$\beta_N = \beta_{N,k(N)}.$$

The uniform price auction satisfies **double largeness** if and only if

$$lim_{N\to\infty} k(N) = \infty,$$

$$lim_{N\to\infty}[N - k(N)] = \infty,$$

$$lim_{N\to\infty} k(N)/N = \alpha \in [0, 1].$$

- Enough goods.
- Enough competition.
- No uncertainty about the ratio in the limit.

Information is aggregated if for each $\omega \in \{\ell, h\}$ and all $\epsilon > 0$

$$\lim_{N\to\infty} Pr(|\beta_N(Y_{k+1:N}) - v_\omega| < \epsilon|\omega) = 1,$$

Proposition 2. *The uniform price auction aggregates information if and only if it satisfies double largeness.*

We consider the case in which

$$\lim_{N\to\infty} \frac{k(N)}{N} = \alpha \in (0, 1)$$

- The case in which $\alpha = 0$ or 1 requires additional work.

Let x_ω solves

$$[1 - G(x_\omega|\omega)] = \alpha.$$

Since $G(1|\ell) = G(1|h) = 1$ and smaller signals are stronger indicators for state ℓ,

$$G(x|\ell) > G(x|h), \forall x \in (0, 1).$$

Therefore,

$$[1 - G(x|\ell)] < [1 - G(x|h)].$$

Hence,

$$x_\ell < x_h.$$

The law of large numbers implies that $Y_{k(N):N} \to x_\omega$ in probability in each state $\omega \in \{\ell, h\}$,

$$\lim_{N\to\infty} Pr(|Y_{k(N):N} - x_\omega| < \epsilon|\omega) = 1, \forall\omega \in \{\ell, h\} \text{ and } \epsilon > 0.$$

Conversely, we have

$$\lim_{N\to\infty} Pr(\omega = \ell|Y_{k(N):N} = x_\ell) = 1.$$

$$\lim_{N\to\infty} Pr(\omega = h|Y_{k(N):N} = x_h) = 1.$$

Therefore,

$$\lim_{N\to\infty} E(v_\omega|Y_{k(N):N} = x_\ell) = v_\ell.$$

$$\lim_{N\to\infty} E(v_\omega|Y_{k(N):N} = x_h) = v_h.$$

Note that

$$\lim_{N\to\infty} \frac{k(N)+1}{N} = \alpha \in (0,1)$$

The law of large numbers also implies that $Y_{k(N)+1:N} \to x_\omega$ in probability in each state $\omega \in \{\ell, h\}$,

$$\lim_{N\to\infty} Pr(|Y_{k(N)+1:N} - x_\omega| < \epsilon|\omega) = 1, \forall\omega \in \{\ell, h\} \text{ and } \epsilon > 0.$$

Therefore,

$$\lim_{N\to\infty} E(v_\omega|Y_{k(N)+1:N} = x_\ell) = v_\ell.$$

$$\lim_{N\to\infty} E(v_\omega|Y_{k(N)+1:N} = x_h) = v_h.$$

In the uniform price auction, the price is $\beta_N(Y_{k(N)+1:N})$.

In state ℓ, when N is large,

$$\beta_N(Y_{k(N)+1:N}) \approx \beta_N(x_\ell).$$

In state h, when N is large,

$$\beta_N(Y_{k(N)+1:N}) \approx \beta_N(x_h).$$

Now, use the "sandwich condition"

$$E(v_\omega|Y_{k(N):N} = x_\ell) \leq \beta_N(x_\ell) \leq E(v_\omega|Y_{k(N)+1:N} = x_\ell).$$

$$E(v_\omega|Y_{k(N):N} = x_h) \leq \beta_N(x_h) \leq E(v_\omega|Y_{k(N)+1:N} = x_h).$$

Hence,

$$\lim_{N\to\infty} \beta_N(x_\ell) = v_\ell$$

$$\lim_{N\to\infty} \beta_N(x_h) = v_h$$

Thus, in each state, the price converges to the true value. Information gest aggregated.

One can easily extend this discrete model (binary state) to a continuous model (continuous state) where $\omega \in [0, 1]$ and v_ω is continuous in ω.

6 Further Discussion

There is an aggregate uncertainty about the signal. That is, the crucial cut-offs x_ℓ and x_h are also random variables.

There is an uncertainty about the relative number of objects, that is, the ratio α is a random variable.

Discriminatory auction, see Jackson and Kremer (2006).

Uncertainty about the number of bidders.

- An informed seller (adverse selection): Lauermann and Wolinsky (2017, 2022).
- Costly entry: Murto and Valimaki (2023).
- Pure randomness: Lauermann and Speit (2022)

Information acquisition, see Atakan and Ekmekci (2023).

Externality of the information, see Atakan and Ekmekci (2014).

All pay auction, see Chi, Murto and Valimaki (2019).

Sequential trading and searching (OTC market), see Lauermann and Wolinsky (2016).

Lecture 2. Linkage Principle

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1 Baisc Setting

We have revenue equivalence for IPV setting for all standard auction forms:

- The expected payoff of the bidder with the lowest signal is 0. (same starting point)
- The bidder with the highest signal wins the auction. (fixed the allocation rule)

What happens if bidders have interdependent values and correlated signals?

For simplicity, consider a two-bidder case with bidders 1 and 2.

Bidder 1 privately knows his signal $x \in [0, 1]$, and bidder 2 privately knows his signal $y \in [0, 1]$.

Correlated signals (x, y) $f(\cdot, \cdot)$. Conditional density $g(y|x)$.

- Contain the independent-signal case where $g(y|x) = g(y)$ for all x and $f(x, y) = g(x)g(y)$.
- Generally with the correlated-signal case, we assume x and y are affiliated, or say, they satisfy MLRP assumption.

Interdependent value, bidder 1 receives $v(x, y)$ when receiving the good while bidder 2 receives $v(y, x)$ when receiving the good.

- Contain private-value case where $v(x, y) = v(x)$.
- Assume $v(x, y) \geq v(y, x)$ if $x \geq y$ and $v(x, y) \leq v(y, x)$ if $x \leq y$.
  - The bidder with a higher signal has a higher valuation. It guarantees that the bidder with the higher signal wins the auction.

Given an auction form and let bidder 2 follow the equilibrium strategy, if the bidder 1 with signal x use the equilibrium strategy of bidder with signal z, let $m(z, x)$ as his expected payment, then his expected payoff is

$$R(z, x) = \int_0^z v(x, y)g(y|x)dy - m(z, x).$$

2 Indepent Signals and Revenue Equivalence

In the IPV Setting, we have

$$R(z,x) = \int_0^z v(x)g(y)dy - m(z).$$

In the equilibrium, it is optimal for each bidder to follow the equilibrium strategy and not to mimic other signals,

$$R_1(x,x) = 0.$$

We then let $V(x)$ as the expected payoff of bidder 1 if his signal is x,

$$V(x) = R(x,x).$$

Envelop formula

$$V'(x) = R_2(x,x) = G(x)v'(x).$$

$$V(x) = V(0) + \int_0^x G(s)v'(s)ds = \int_0^x G(s)v'(s)ds$$

Hence, bidders with the same signal receive the same expected payoff from all standard auction formats.

$$m(x) = G(x)v(x) - V(x).$$

Thus, bidders with the same signal make the same expected payment, and revenue equivalence holds

What happens for interdependent value and independent signals?

$$R(z,x) = \int_0^z v(x,y)g(y)dy - m(z).$$

We still have the first-order condition,

$$R_1(x,x) = 0.$$

We still have the envelope form

$$V'(x) = R_2(x,x) = \int_0^x v_1(x,y)g(y)dy$$

$$V(x) = V(0) + \int_0^x \int_0^s v_1(s,y)g(y)dyds.$$

Bidders with the same signal still receive the same expected payoff from all standard auction formats.

$$m(x) = G(x)v(x) - V(x).$$

Thus, bidders with the same signal make the same expected payment, and revenue equivalence holds.

For independent-signal case, the bidder's expected payment $m(z,x)$ is independent of his own signal x and only depends on his report z,

$$m(z,x) = m(z), \forall z, x.$$

We therefore can get rid of $m(x)$ by using the envelop form, showing that the bidders' expected payoffs and payments are uniquely pinned down by the allocation rule.

3 Correlated Signals and Linkage Principle

Things become different when we look at the case if signals are correlated. For simplicity, we consider the private value case.

$$R(z,x) = \int_0^z v(x)g(y|x)dy - m(z,x).$$

Now, even if bidder 1 with signal x bids the same as if his signal were z, he makes a different expected payment.

- For the first price auction,

$$m(z,x) = G(z|x)b_1(z).$$

- For the second price auction,

$$m(z,x) = G(z|x)E[b_2(y)|x, y \leq z]$$

  where,

$$E[b_2(y)|x, y \leq z] = \int_0^z b_2(y) \cdot \frac{g(y|x)}{G(z|x)}dy = \int_0^z b_2(y)d\left[\frac{G(y|x)}{G(z|x)}\right].$$

Assume $x > z$. If the bidder 1 with signal x bids the same as the bidder with signal z, in the first price auction, conditional on winning, he always pays $b_1(z)$, which is independent of his signal. However, in the second price auction, he has to pay more conditional on winning,

$$E(b_2(y)|x, y < z) \geq E(b_2(y)|z, y < z),$$

Or say,

$$\int_0^z b_2(y)d\left[\frac{G(y|x)}{G(z|x)}\right] \geq \int_0^z b_2(y)d\left[\frac{G(y|z)}{G(z|z)}\right].$$

This is because (1) $b_2(y)$ is an increasing function and

$$\frac{G(y|x)}{G(z|x)} \quad FOSD \quad \frac{G(y|z)}{G(z|z)}.$$

FOSD is equivalent to

$$\frac{G(y|x)}{G(z|x)} \leq \frac{G(y|z)}{G(z|z)}, \forall y \in [0,z],$$

Which is equivalent to

$$\frac{G(y|x)}{G(y|z)} \leq \frac{G(z|x)}{G(z|z)} = \frac{G(y|x) + G(z|x) - G(y|x)}{G(y|z) + G(z|z) - G(y|z)}, \forall y \in [0,z].$$

We then only need to prove

$$\frac{G(y|x)}{G(y|z)} \leq \frac{G(z|x) - G(y|x)}{G(z|z) - G(y|z)}$$

From the MLEP assumption, we have

$$\frac{G(y|x)}{G(y|z)} = \frac{\int_0^y g(s|x)ds}{\int_0^y g(s|z)ds} \leq \frac{g(y|x)}{g(y|z)},$$

$$\frac{G(z|x) - G(y|x)}{G(z|z) - G(y|z)} = \frac{\int_y^z g(s|x)ds}{\int_y^z g(s|z)ds} \geq \frac{g(y|x)}{g(y|z)}.$$

Therefore, compared to the first price auction, in the second price auction, it is more costly for the bidder 1 with signal x to lie that his signal is $z < x$. Hence, the seller needs to pay a smaller amount of information rent to him in the second price auction. An intuitive guess is that the seller receives higher revenue in the second price auction than in the first price auction.

We now state the intuition properly as the linkage principle. First, denote $W(z, x)$ as the expected payment conditional on (1) winning the auction, (2) receiving signal x, (3) biding as if the signal were z. We have

$$m(z, x) = G(z|x)W(x, x)$$

$$W^{FPA}(z, x) = b_1(z),$$

$$W^{SPA}(z, x) = E[b_2(y)|x, y \leq z].$$

Proposition 1. *Standard auction A yields higher revenue than standard auction B if*

- $W_2^A(x, x) \geq W_2^B(x, x), \forall x,$
- $W^A(0, 0) = 0 = W^A(0, 0).$

Note that the $W(x, x)$ is the bidder's equilibrium payment conditional on winning given that his signal is x. The first condition claims that the revenue is higher if his equilibrium payment conditional on winning is more sensitive to his own signal.

Proof. We have

$$R^A(z, x) = \int_0^z v(x)g(y|x)dy - G(z|x)W^A(z, x).$$

The first-order condition

$$R_1(x, x) = 0.$$

We have

$$R_1(z, x) = v(x)g(z|x) - g(z|x)W^A(z, x) - G(z|x)W_1^A(z, x).$$

Hence,

$$R_1(x, x) = v(x)g(x|x) - g(x|x)W^A(x, x) - G(x|x)W_1^A(x, x) = 0,$$

$$W_1^A(x, x) = \frac{g(x|x)}{G(x|x)}v(x) - \frac{g(x|x)}{G(x|x)}W^A(x).$$

Also for auction B

$$W_1^B(x, x) = \frac{g(x|x)}{G(x|x)}v(x) - \frac{g(x|x)}{G(x|x)}W^B(x).$$

Take the difference,

$$W_1^A(x, x) - W_1^B(x, x) = -\frac{g(x|x)}{G(x|x)}[W^A(x, x) - W^B(x, x)]$$

Let

$$\Delta(x) = W^A(x,x) - W^B(x,x).$$

We would like to show that $\Delta(x) \geq 0$ for any x. Hence, the bidder pays more in auction A no matter what his signal is, which completes the proof. Now since

$$\Delta'(x) = W_1^A(x,x) - W_1^B(x,x) + W_2^A(x,x) - W_2^B(x,x)$$

Therefore,

$$\Delta'(x) = - - \frac{g(x|x)}{G(x|x)}\Delta(x) + [W_2^A(x,x) - W_2^B(x,x)].$$

The second term is non-negative, we hence have

$$\Delta'(x) \geq 0 \text{ when } \Delta(x) \leq 0,$$

which is sufficient to show that $\Delta(x)$ is non-negative. □

Note that we can weaken the first condition by

$$W_2^A(x,x) \geq W_2^B(x,x) \text{ when } W_2^A(x,x) = W_2^B(x,x)$$

In the first price auction

$$W_2^{FPA}(z,x) = \frac{d}{dx}b_1(z) = 0,$$

while in the second price auction

$$W_2^{SPA}(z,x) \geq 0$$

Therefore, the second price auction generates a higher revenue for the seller than the first price auction

4 Public Information

The seller may have information that is potentially useful to the bidders.

Should the seller keep it hidden or should she reveal it publicly?

We still focus on the simplest case, two bidders, private value, and correlated signals.

Let $s \in (0,1)$ be a random variable that denotes the information available to the seller.

Bidder 1's payoff: $v(s,x,y)$.

Bidder 2's payoff: $v(s,y,x)$.

The payoff v is increasing in s.

Deote

$$v(x,y) = E_s(v(s,x,y)).$$

The information s is affiliated with x and y (MLRP).

Consider the first price auction

If the seller does not release the public information, as shown before

$$W_2^N(z, x) = 0.$$

If the seller releases the information s, the bidder 1's expected payment conditional on winning is

$$W^F(z, x) = \int_0^1 b(s, z) f(s|x) ds$$

Note that $b(s, z)$ is increasing in s and since s and x is affiliated,

$$F(s|x) \text{ FOSD } F(s|x') \text{ when } x > x'.$$

Hence,

$$W_2^F(z, x) > 0.$$

By the linkage principle, releasing the the public information increases the seller's revenue

5 Further Extention

One can easily extend the proof to the case with more than two bidders.

- Bidder 1 versus the bidder receiving the highest signal among other bidders.

English auction

- Equivalent to the second price auction in the private value setting, drop out at $v(x)$.
- For correlated value, one has to first specify the symmetric equilibrium.
- One then can show that the English auction generates more revenue for the seller than the second price auction.
- See Chapter 6 of Auction Theory by VJ Krishna.
- Try to understand that through the linkage principle.

6 Information Acquisition

Until now, bidders' signals or say, their signals are exogenously given.

What happens if bidders need to acquire the information at some cost?

The bidder i's signal is θ_i, he can acquire a signal s_i with a distribution $F(s_i|\theta_i)$. Different distribution F comes with different cost

Persico (2001), the bidders have more incentive to acquire information under the first price auction instead of the second price auction.

Hence, if we take into account the information acquisition, the first price auction might generate more revenue for the seller than the second price auction.

- The bidders' valuations are more dispersed. Note that the max function is convex.

Bobkova (2024): $\theta_i = v + \epsilon_i$, the v is a common-value term, same for all the bidders while ϵ_i is a private value term, which is drawn independently.

- Bidders have more incentive to learn the common-value term under the first price auction, while they have the same incentive to learn the private-value term in the first and the second price auctions.
- Therefore, in the first price auction, the bidders will spend more resources on learning the common-value term while spending fewer resources learning the private-value term.
- However, for efficient allocation, only the value of the private-value term matters.
- The second-price auction is more efficient than the first-price auction.

Lecture 3. First-Price Auctions With General Information Structures

1 Introduction

- First-price auction.
- Private-value case: IPV setting.
- Interdependet value case: common value setting with conditional independent signals.
- General setting with affiliated signals.
- We always assume that each bidder can observe his own signal.
- Strong assumptions on the information structures and the bidder's beliefs about others' information.
- Bergmann, Brooks, and Morris (ECTA 2017): characterize the set of equilibrium outcomes in the first-price auction under all possible common-prior information structures.
  - The distribution $G(s_1, ..., s_n, v_1, ..., v_n)$.
- Identify the lower bound of the distribution of winning bids.
  - FOSD
  - A lower bound of the seller's expected surplus.
  - Exact information structure and equilibrium in the symmetric case, which is also efficient.
  - A upper bound of the bidders' expected surplus.
- Nice writing and beautiful proof.

2 Common-Value Setting

- Based on Ben Brooks's talk.

- N bidders.

- Pure common value v that is drawn from the c.d.f P on the support $[\underline{v}, \bar{v}]$.
  - Common prior.

- The players submit bids $b_i \in \mathbb{R}_+$.

- The winner is chosen randomly from the set of high bidders, the winner must pay his bid.
  - No reserve price, always allocate the goods.
  - Seller receives no value when keeping the goods.

- The information structure is not specified.
  - The distribution $G(s_1, ..., s_n, v)$

- The total surplus is always:
$$\hat{v} = \int_{v=\underline{v}}^{\bar{v}} v dP(v).$$

- The split between the seller and the bidders depends on the information structure and the selection of the equilibrium.

- Information structure and the equilibrium generates the maximum revenue: all bidders have the same information, competing with each other.
  - The affiliation between bidders' information is maximized.
  - The seller has the full surplus.

- What is the minimum revenue?
  - "Minimum" winning bid distribution $\underline{H}(b)$.
  - FOSD: For each $H(b)$ induced by an information structure and an equilibrium, we have $\underline{H}(b) \geq H(b)$ for any b.

- Now, what conditions must $H(b)$ satisfy if it is induced by an information structure and an equilibrium?
  - There are many conditions. But only one condition is important and binding when we need to find the lower bound

- A given information structure and equilibrium under it induces a joint distribution between the true value v and the winning bid b, denoted by $H(b|v)$.

- A joint distribution between the true value v, the winning bid b, and the winner's identity i, denoted by $H_i(b|v)$

$$H(b|v) = \sum_i H_i(b|v).$$

- Bidder i's equilibrium surplus is

$$U_i = \int_{v=\underline{v}}^{\bar{v}} \int_{x=0}^{\infty} (v-b) dH_i(b|v) dP(v).$$

- Given the winning bid distribution (including the joint distributions), bidders must prefer not to deviate. Let us focus on a special deviation.

- Consider **uniform upward deviation**:
  - Bid b^* whenever you would have bid $x \leq b^*$ in equilibrium;
  - If you would have bid $x > b^*$, do not change your bid.
  - Remark: for the uniform upward deviation, bidders disregard their own private information, only conditional on whether he losses or wins the auction before the deviation.

- The expected payoff of bidder i after an uniform upward deviation to b is denoted by $V_i(b^*)$.

- Instead of focusing on the lower bound of the winning bid distributions induced by some information structure and equilibrium, consider a relaxed problem and focus on the winning bid distribution satisfying $V_i(b^*) \leq U_i$ for each i and b^*, that is, the difference in payoffs $V_i(b^*) - U_i \leq 0$.

- Now, let us calculate the difference in payoffs.

- Since the prior is symmetric, by a permutation argument (see Lemma 3 of BBM), it is without generality to focus on the symmetric case:

$$H_i(b|v) = \frac{1}{N} H(b|v).$$

- Hence, the bidder i losses the auction with probability $(N-1))/N$ and wins the auction with probability $1/N$.

- Conditional on losing the auction, the deviation changes bidders i's payoff if the winning bid b is less than b^*. In this case, bidder i's payoff changes from 0 to $E(v|b, loss) - b^*$, which is $E(v|b) - b^*$ in this common-value setting.

- Conditional on winning the auction, the deviation changes bidders i's payoff if the winning bid b is less than b^*. In this case, bidder i needs to pay additionally $b^* - b$ which is independent of his own value.

- Remark: conditional on b, the realized value v only matters when this bidder loses in the equilibrium.

- Therefore, we can rewrite $V_i(b^*) - U_i \leq 0$ by:

$$\frac{N-1}{N} \int_0^{b^*} [E(v|b) - b^*] \, dH(b) \leq \frac{1}{N} \int_0^{b^*} (b^* - b) dH(b). \tag{1}$$

- Note that this condition is most slack if $E(v|b)$ is minimized pointwise for each b, by making b and v perfectly correlated, that is, the winning bid b is a deterministic and strictly increasing function of v:

$$\beta(v) = min\{b|H(b) \geq P(v)\}.$$

  - The losers have more incentive to deviate since the conditional expected value of v is higher when v is non-monotonic in the winning bid.

- Therefore, it is without loss of generality to focus on the case where the winning bid b is a deterministic and strictly increasing function of v.

- The constraint (1) is equivalent to

$$\frac{N-1}{N} \int_{v=\underline{v}}^{v^*} (v - \beta(v^*)) dP(v) \leq \frac{1}{N} \int_{v=\underline{v}}^{v^*} [\beta(v^*) - \beta(v)] dP(v). \tag{2}$$

  where

$$v^* = max\{v|\beta(v) \geq b^*\}.$$

  - Mimic the winner with value v^* if I bid lower than him.

- Now try to minimize $\beta(v)$ satisfying (2).

- Rewrite (2) as:

$$\beta \geq \Lambda(\beta)$$

  where

$$\Lambda(\beta)(v) = \frac{1}{P(v)} \int_{x=\underline{v}}^{v} \left(\frac{N-1}{N} x + \frac{1}{N} \beta(x) \right) dP(x). \tag{3}$$

- Note that:

  - If β satisfies (2), then $\Lambda(\beta)$ also does.

- $\Lambda(\beta)(v) \leq \beta(v)$ for all v.
- Contraction: $\max_v |\Lambda(\beta)(v) - \Lambda(\hat{\beta})(v)| \leq \frac{1}{N} \max_v |\beta(v) - \hat{\beta}(v)|$

- Hence, starting from any feasible β^0, the sequence of functions β^k defined by $\beta^k = \Lambda(\beta^k - 1)$ is feasible, converging pointwise to the unique fix point

$$\underline{\beta}(v) = \frac{1}{P^{(N-1)/N}(v)} \int_{x=\underline{v}}^{v} x \frac{N-1}{N} \frac{dP(x)}{P^{1/N}(x)},$$

which generates $\underline{H}(b)$, the lower bound of the winning bid distribution satisfying (2).

  - the upper bound of the bidders' expected surplus since the outcome is efficient.

- Note that we consider a relaxed problem, we need to verify that $\underline{H}(b)$ is a tight bound if there exists an information structure and an equilibrium generates it.

- Information structure: One bidder receives signal $s = v$ and other bidders' signals are independent draws from the conditional distribution $F(s)/F(v)$ where $F(v) = (P(v))^{1/N}$.
  - Bidders' signals are i.i.d. The distribution is $F(s)$.
  - Remark: each bidder's signal is not independent of the highest signals among them.
  - Minimize the affiliation among bidders' signals.

- Equilibrium: the bidder with signal s bids $\underline{\beta}(s)$.
  - Bidders are indifferent between any upward deviation.
  - Remark: equivalent to indifferent to any uniform upward deviation.

- Why?
  - If a bidder with a lower signal strictly prefers not to mimic the bidder with a higher signal, it means that the bidder with the higher signal pays "too much" to the seller conditional on winning. Hence, there is a room to move the winning bid distribution lower.
  - The seller's revenue is pinned down by the incentive constraint of the upward deviation.
  - When solving the optimal mechanism, we usually let the incentive constraint of the downward deviation binding and the incentive constraint of the upward deviation slack. Here, we have a reverse goal. (Duality?)

3 General Setting

- Each bidder i has a value $v_i \in [\underline{v}, \bar{v}]$.

- Values $\mathbf{v} = \{v_1, v_2, ..., v_N\}$ are jointly distributed according to μ.
  - Symmetric common prior.

- We now follow the steps in Section 2. Now if the bidder i prefers not to do a uniform upward deviation to b^*:

$$\frac{N-1}{N}\int_0^{b^*}\left[E(v_i|b, loss) - b^*\right] dH(b) \leq \frac{1}{N}\int_0^{b^*}(b^* - b)dH(b). \tag{4}$$

- Only need to focus on the case where the outcome is efficient, that is, the bidder with the highest value wins the auction.
  - Depress $E(v_i|b, loss)$ the most.

- Define

$$\alpha(\mathbf{v}) = E(v_i|\left[\mathbf{v}\right], loss) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - v_{max}\right).$$

- Since whether i loses is independent of the realizations of the winning bid, we can rewrite (4) as:

$$\frac{N-1}{N}\int_0^{b^*}\left[E(\alpha(\mathbf{v})|b) - b^*\right] dH(b) \leq \frac{1}{N}\int_0^{b^*}(b^* - b)dH(b).$$

- Hence, we can only focus on $\alpha(\mathbf{v})$ instead of the whole $\mathbf{v}$. Let $\omega = \alpha(\mathbf{v})$ and we now return to the analysis in Section 2, replacing the distribution P by Q, the distribution of ω.
  - When focusing on upward deviation, only the value conditional on losing matters.

- Information structure: the bidder with the highest signal receives the signal $s = \omega$, other bidders receive signals which are independent draws from the conditional distribution $F(s)/F(\omega)$ with $F(s) = (Q(s))^{1/N}$.
  - Bidders' signals are i.i.d. The distribution is $F(s)$.

- Equilibrium: each bidder bids $\underline{\beta}(s)$.

- Now, we consider the upper bound of the revenue.

- First, consider the tie-breaking rule in favor of the bidder with the highest value. In this case, we can let every bidder receives a signal equal to the highest value, bid his signal, and then let the bidder with the highest value win.
  - Maximize the affiliation and competition.
  - Seller receives full revenue while bidders receive 0.

- It is possible to achieve approximately the same outcome with the uniform tie-breaking rule.
  - Make the signals highly affiliated.
  - The bidders not receiving the highest value always lose. Hence, they will stay with high bids and pushes the bidder with the highest value to bid close to his value.

- The lower bound of total surplus.
  - A highly inefficient equilibrium.
  - The information is "reversed".

- What happens if the prior is not symmetric?
  - The lower bound is not tight anymore.
  - A permutation result

- What happens if each bidder can observe his own value?

- Let us start with the lower bound of revenue first.
  - We can still focus on the efficient outcome. Hence, we also have the upper bound of bidders' surplus
  - Competition between bidders increases, which decreases the bidders' surplus and increases the seller's revenue.
  - The incentive constraint for upward deviation is tighter since before we consider the bidder with average value conditional losing. Now we have to consider the bidder with the highest value among all bidders lose.

- For the upper bound of revenue, it decreases since bidders have some private information and receive information rents.

- For the lower bound of bidders' surplus, it increases since the competition between bidders and private information rules out many inefficient equilibria.

- Example with independent value.

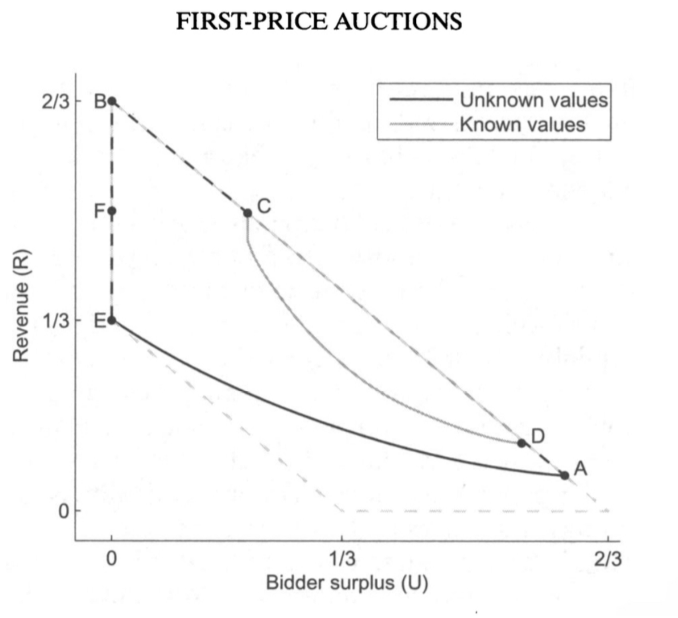

Figure 1: Indepent-Value Case

4 Related Work

- Du (ECTA, 2018).
  - The mechanism in the common-value setting guarantees the revenue of the seller converges to full surplus as the number of bidders grows large, under any information structure.
- Brooks and Du (ECTA, 2021).
  - The optimal mechanism for any finite number of bidders in the common-value setting when the information structure is not specified.
- Brooks and Du (2023).
  - Robust mechanism financing public goods under interdependent value setting.
- What about voting?
  - Under a common-value setting, the information is aggregated and voters reach their first-best outcome under any qualified majority rule when the number of voters grows large.
  - However, this result is fragile when there is an uncertainty in the information structure (Mandler, GEB 2012) and misspecification (Ellis, TE 2016).
- Auctions with limited information.

- The seller has limited information about the distribution of bidders' valuation, see the work by Wanchang Zhang from UCSD.
- The bidders have limited information about the other bidder's valuation or cannot perfectly anticipate the other bidders' bidding behavior, see the work by Bernhard Kasberger from Heinrich Heine University.