Practical Quantum Computing

Lecture 02

Gates and Circuits

Week	**Tuesday** (3h)			**Wednesday** (3h)			**Deadlines**	
1. The Basics	Introduction	Gates	Circuit Identities	Qiskit	Cirq/Qualtran	Q&A		
	Programming Assignment 1: The basics of a quantum circuit simulator			**Programming Assignment 1:** The building blocks of a quantum circuit simulator				
2. Entanglement and its Applications	Teleportation	Superdense Coding	Quantum Key Distribution	PennyLane	Terminology of Projects	Q&A		
	Programming Assignment 2: The basics of a quantum circuit optimizer			**Programming Assignment 2:** The building blocks of a quantum circuit optimizer				
3. Computing	Phase Kickback and Toffoli	Distinguishing quantum states and The First Algorithms	Grover's Algorithm	Invited TBA		Q&A		11 May 2024
4. Advanced Topics*	Arithmetic Circuits*	Fault-Tolerance*	QML*	Invited TBA	Crumble	Q&A	18 May 2024	

* not evaluated

Learning goals - 02 Gates (The Basics)

1. What you have learned by now
   a. Quantum software: what, why and how
   b. Quantum circuits: diagrams and difficulty of classical simulation
2. **The math behind quantum circuits**
   a. Qubit states are complex vectors
   b. Quantum gates are complex matrices
   c. Computations are matrix vector multiplications
   d. Gates are rotations of vectors in a complex space
   e. The tensor product for building larger vectors and matrices
3. **Types of quantum gates**
   a. Single qubit gates
   b. Two qubit gates
   c. Gates are rotations of vectors in a complex space

In the exercise session and programming assignment of this week

- basics of quantum circuit simulator
- build our own quantum circuit simulator

Computational Basis

We normally expand the wavefunction in terms of a basis of bit strings: the **computational basis**, aka the Z basis.

$$|\psi\rangle = \frac{i}{\sqrt{3}}|010\rangle + \frac{\sqrt{2}}{\sqrt{3}}|111\rangle$$

2^n amplitudes for n qubits.

Other bases are sometimes convenient, e.g., the X basis

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

States as Column Vectors

$$|0\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha|0\rangle + \beta|1\rangle \leftrightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

States as Column Vectors

$$|00\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad |01\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

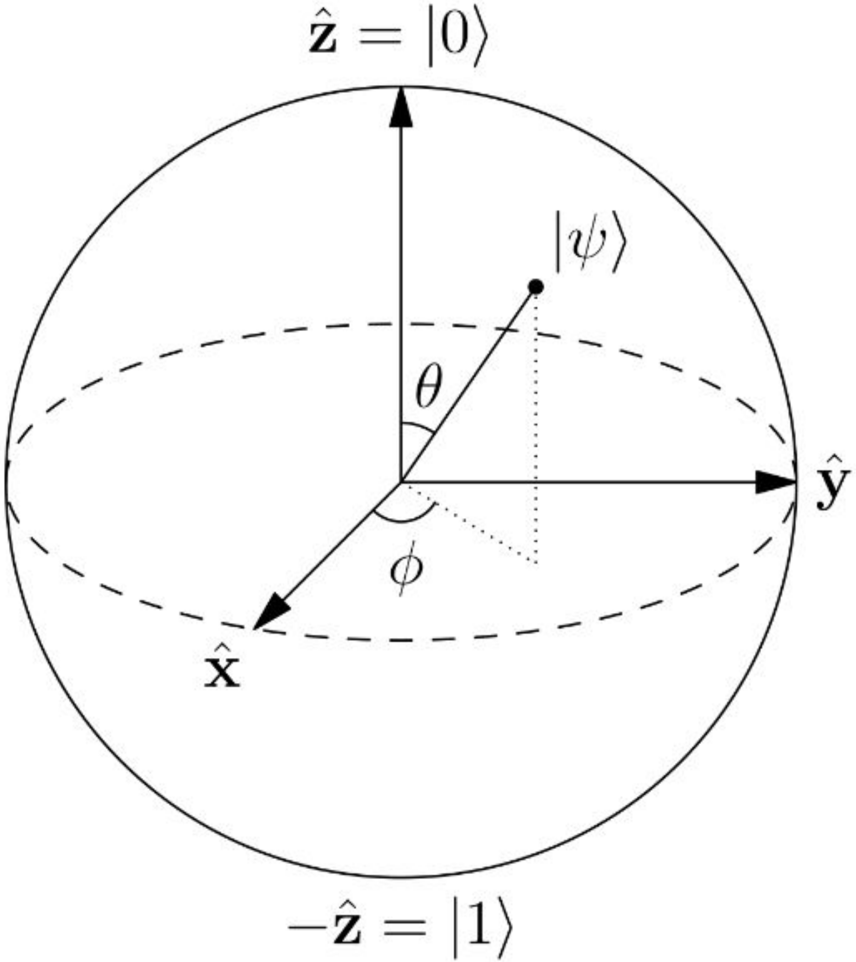
$$|10\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad |11\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Bloch Sphere

Antipodal points = **orthogonal** states (perfectly distinguishable)

Rotations = **unitary** operations

No convenient analogue for multiple qubits, but still useful for a single qubit



Measurement and Born Rule

Quantum state is not directly observable --- sampling from the Born distribution is all we can do.

Quantum computer outputs 1s and 0s.

Probability = Absolute Value Squared of Amplitude.

Repeating a measurement **immediately** returns the **same** answer.

Must repeat the whole experiment to resample from the distribution.

Expectation Values

The averages of quantities can also be calculated from the wavefunction.

Expectation values are not directly observable: only recoverable after many measurements as the mean.

Consequence of the Born rule, not a separate axiom.

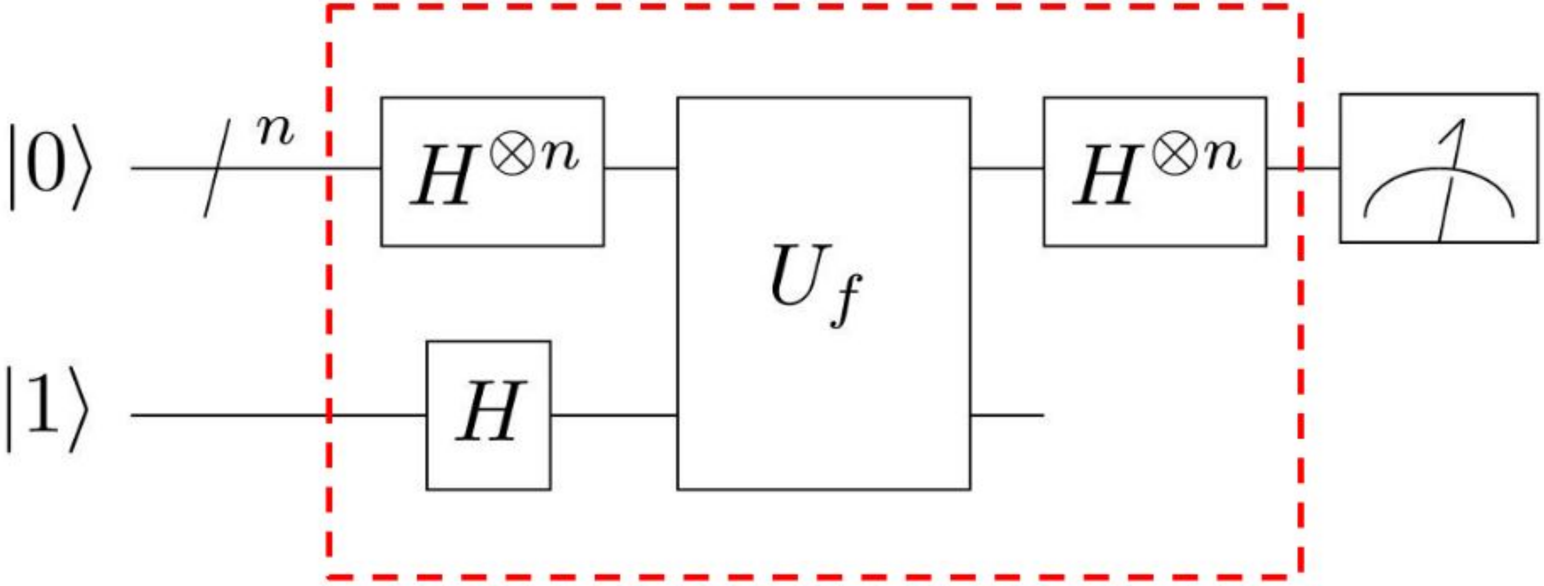
$$\langle X \rangle = \langle \psi | X | \psi \rangle = \text{ Expectation value}$$

Don’t be afraid of this notation. I will explain into more detail in the next lecture

Quantum Circuit = Time Evolution

We construct the time evolution operator from simple building blocks.

Those building blocks are the quantum gates.



Operators as Matrices

Just like how we represent states as column vectors, we can represent operators as matrices which act on those column vectors.

For example, the X operator: $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli-X (NOT)

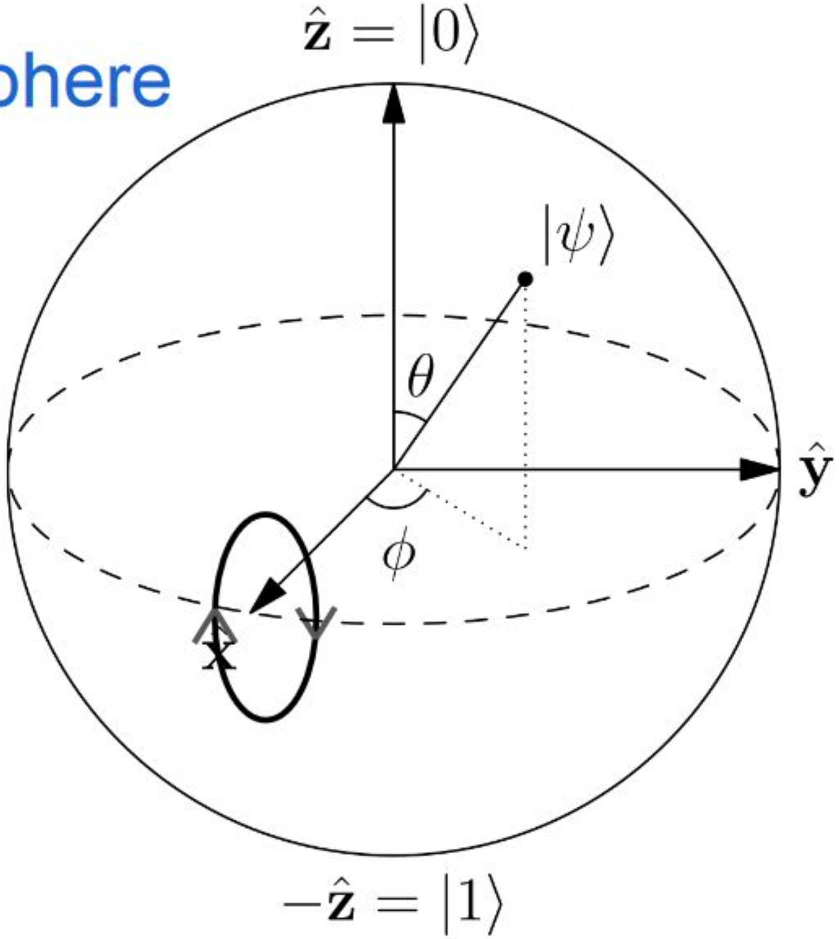
$$|0\rangle \mapsto |1\rangle$$
$$|1\rangle \mapsto |0\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

X Operator on the Bloch Sphere

Rotates around the X axis by 180°

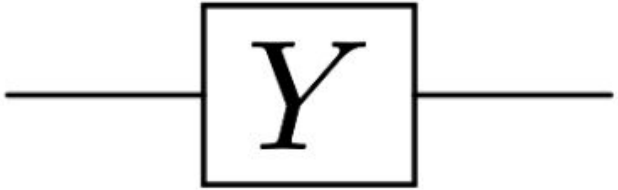
Clockwise or counterclockwise?



Pauli-Y

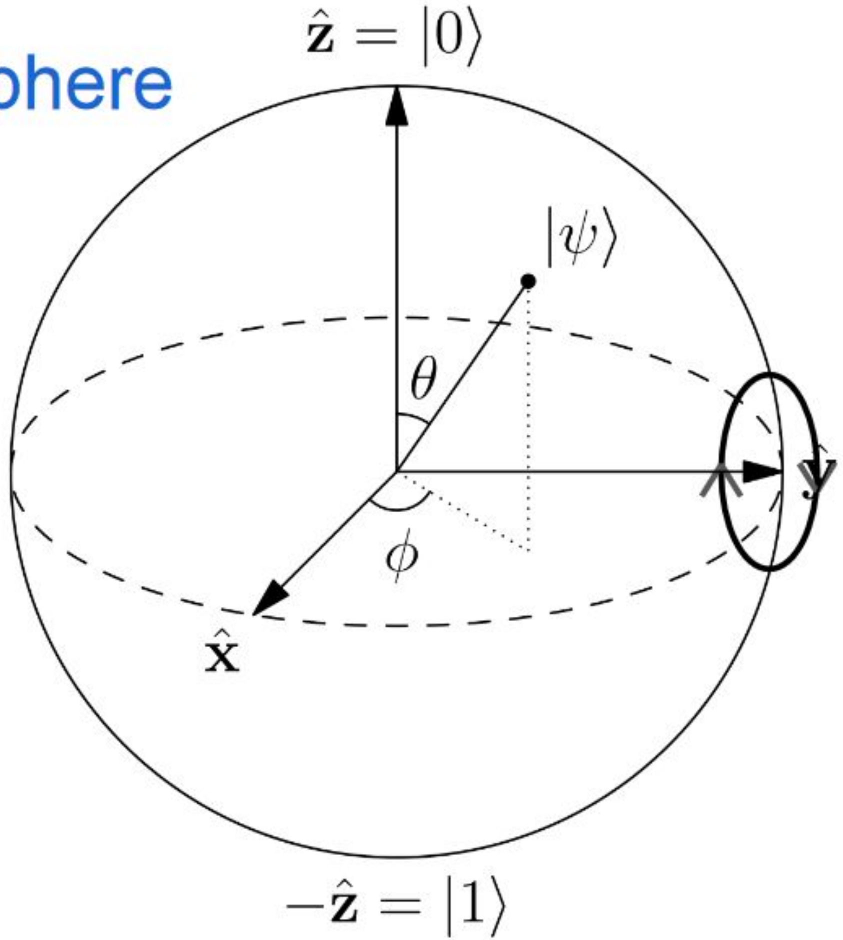
$$|0\rangle \mapsto i|1\rangle$$
$$|1\rangle \mapsto -i|0\rangle$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



Y Operator on the Bloch Sphere

Rotates around the Y axis by 180°



Pauli-Z (Phase Flip)

Diagonal in the computational basis

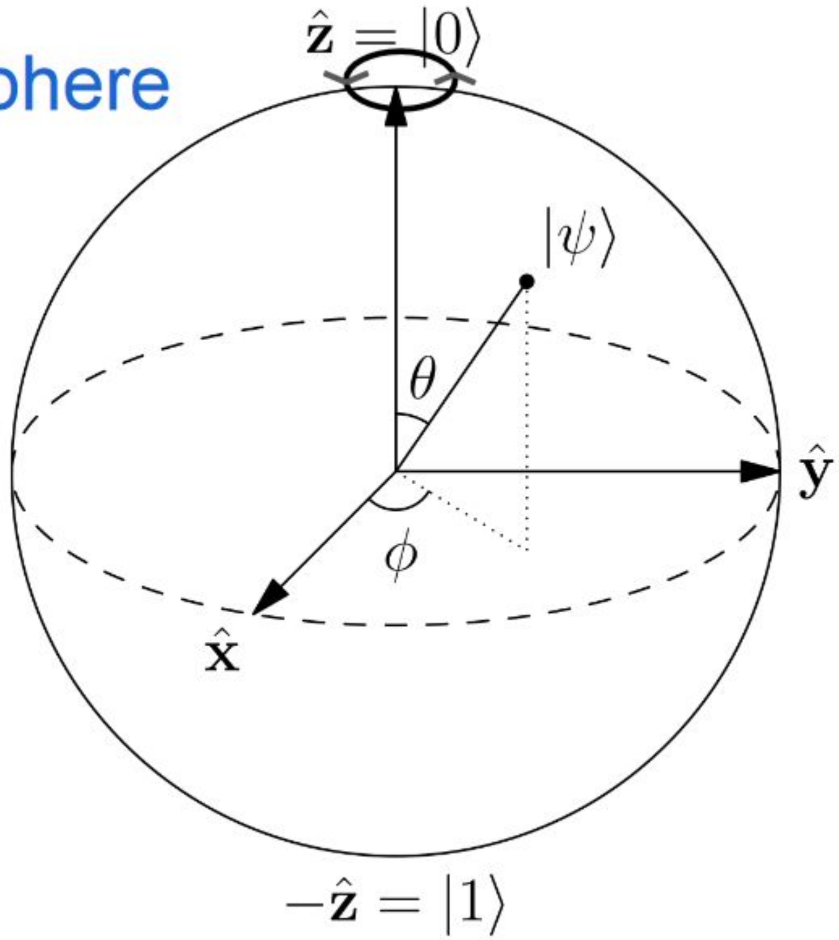
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|0\rangle \mapsto |0\rangle$$

$$|1\rangle \mapsto -|1\rangle$$

Z Operator on the Bloch Sphere

Rotates around the Z axis by 180°



Hadamard Gate

$$|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

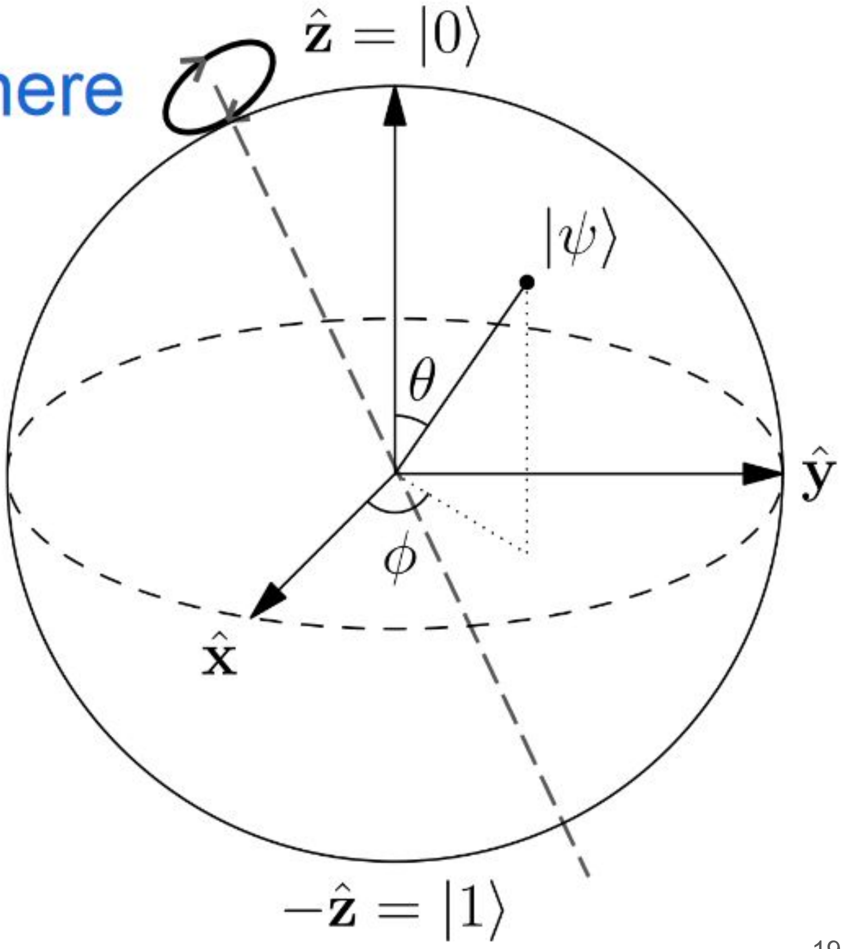
$$H = \frac{1}{\sqrt{2}}(X + Z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

H Operator on the Bloch Sphere

Rotates around the “X+Z” axis by 180°

Exchanges X with Z

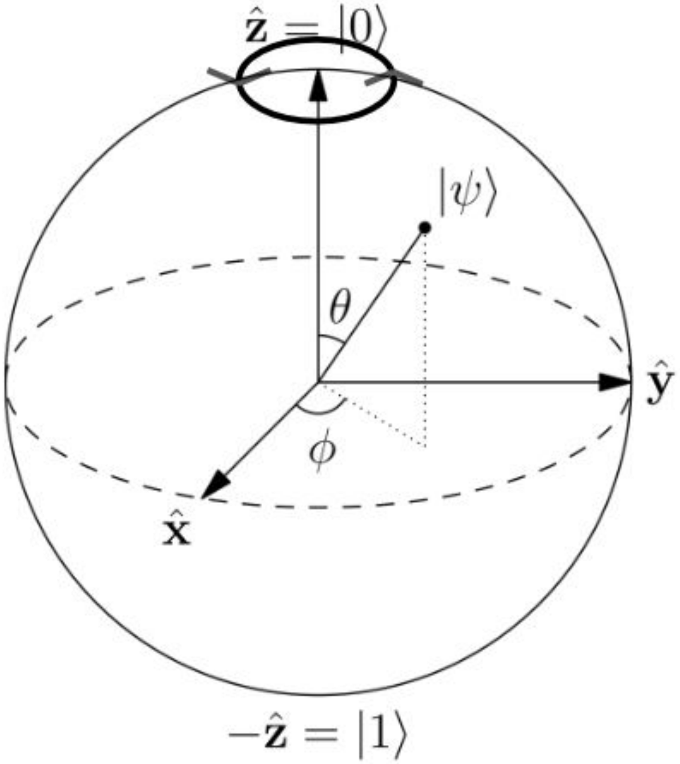
$$HZH = X$$



Arbitrary Rotations of the Bloch Sphere

Exponentiate the Z operator to rotate by an arbitrary angle around the Z axis.

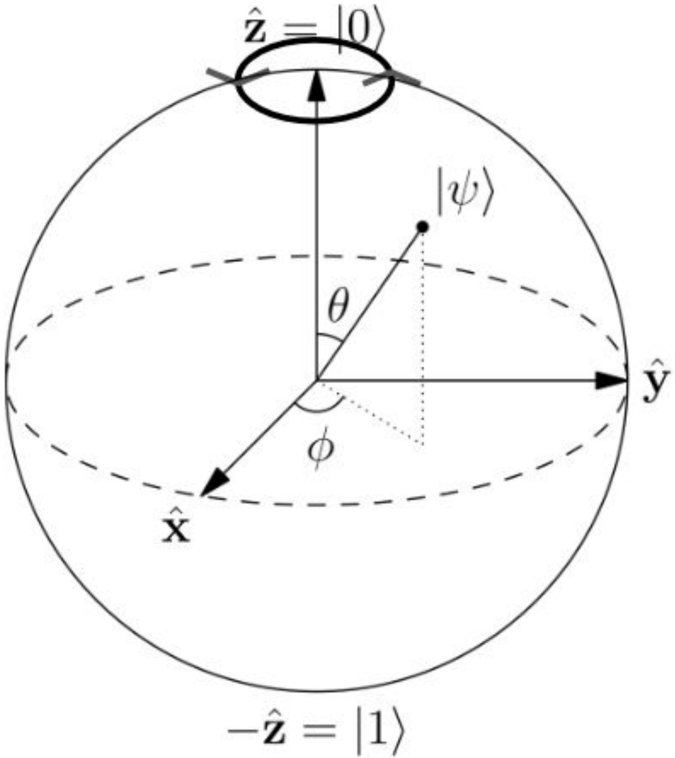
$$e^{-i\pi x Z} = \begin{pmatrix} e^{-i\pi x} & 0 \\ 0 & e^{i\pi x} \end{pmatrix}$$



Arbitrary Rotations of the Bloch Sphere

What about a ½ rotation? Shouldn't that just be Z?

$$e^{-i\pi Z/2} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$



Arbitrary Rotations of the Bloch Sphere

Quarter-rotation and eighth-rotation have names (up to overall phase).

$$
\begin{aligned}
e^{-i\pi Z/4} &= \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \\
&= e^{-i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \\
&= e^{-i\pi/4} S
\end{aligned}
$$

$$
\begin{aligned}
e^{-i\pi Z/8} &= \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} \\
&= e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \\
&= e^{-i\pi/8} T
\end{aligned}
$$

Arbitrary Rotations of the Bloch Sphere

Trick for exponentiating certain operators. Works because $Z^2 = 1$.

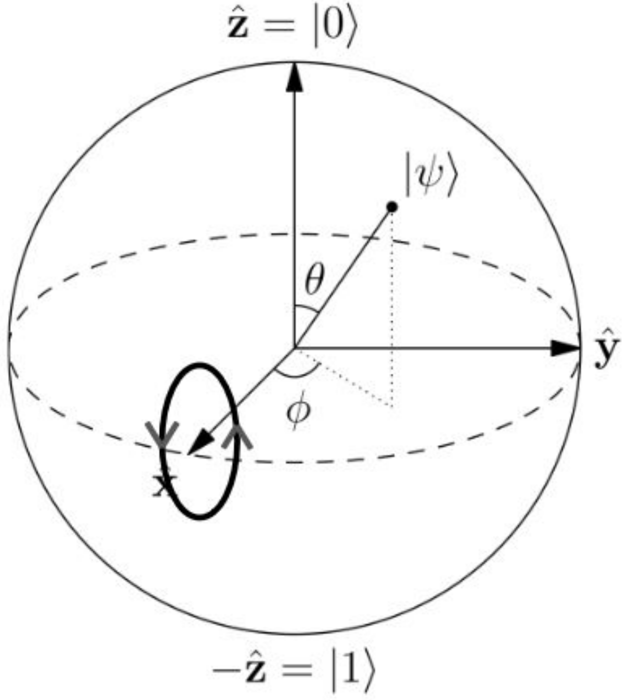
Similar formula for other Pauli matrices.

$$\begin{aligned} e^{-i\pi x Z} &= cos\pi x - i(\sin \pi x)Z \\ &= \begin{pmatrix} e^{-i\pi x} & 0 \\ 0 & e^{i\pi x} \end{pmatrix} \end{aligned}$$

Arbitrary Rotations of the Bloch Sphere

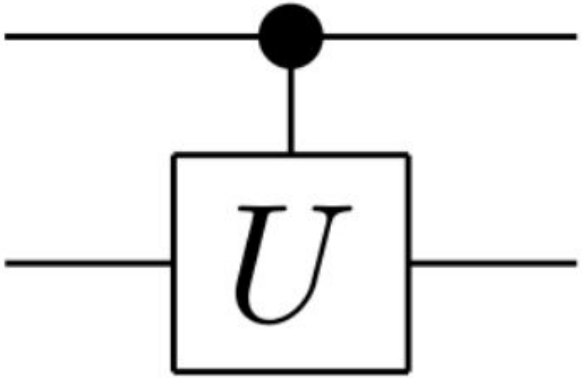
Exponentiate the X operator to rotate by an arbitrary angle around the X axis, similarly with Y

$$e^{-i\pi x X} = \begin{pmatrix} \cos \pi x & -i \sin \pi x \\ -i \sin \pi x & \cos \pi x \end{pmatrix}$$



Controlled Gates

Acts as unitary operator U on the target qubit when the control qubit is in the |1> state.



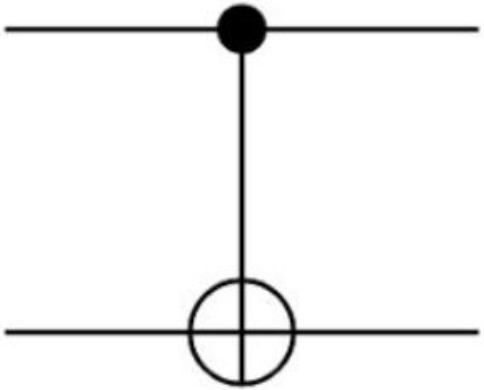
$$\mathrm{CU} = \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{U} \\ 0 & 0 & & \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

Controlled NOT (CNOT)

If the control bit is |0>, the target bit is left unchanged.

If the control bit is |1> then the target bit is flipped.

$$\mathrm{CNOT} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

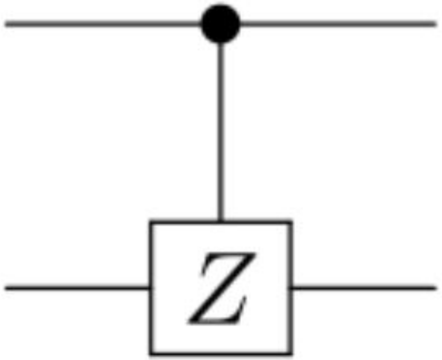


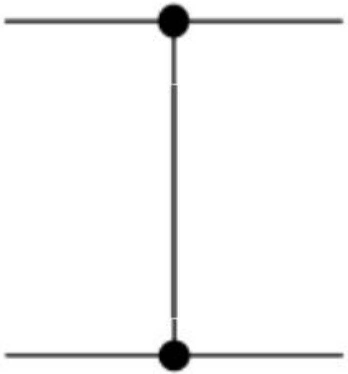
Controlled Z (CZ)

Acts as Z on the target qubit when the control bit is |1>.

CZ is symmetric between the two qubits --- it doesn't matter which bit is the control!

$$
\mathrm{CZ} = \begin{array}{c c}
\begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}
\end{array}
$$





Controlled Rotation

Acts as Z rotation on the target qubit when the control bit is |1>.

How does this compare with CZ?

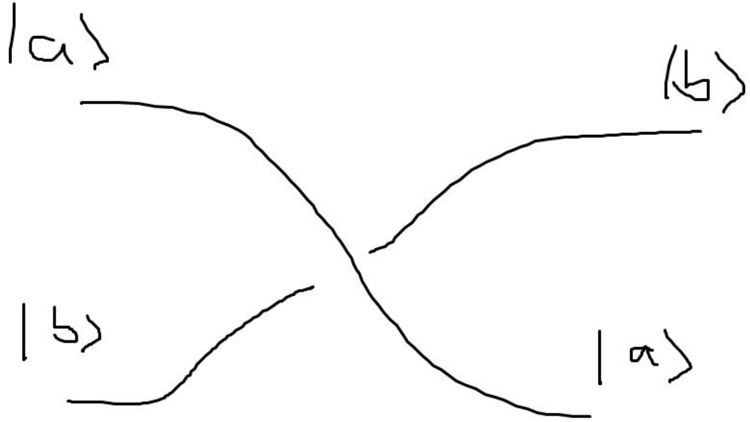
$$
\mathrm{CR} = \begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}
$$

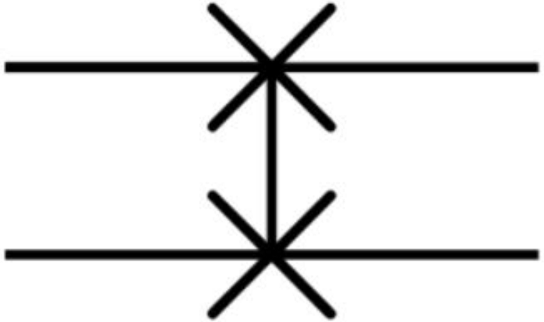
SWAP

Exchanges the states of two qubits.

Equivalent to “crossing the wires.”

$$\mathrm{SWAP} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} & \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$





Tensor Product Gates

$$Z \otimes I = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$I \otimes Z = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$Z \otimes Z = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}$$

Tensor Product Gates

$$Z \otimes I \otimes I \qquad \mathrm{diag}(+1, +1, +1, +1, -1, -1, -1, -1)$$

$$I \otimes Z \otimes I \qquad \mathrm{diag}(+1, +1, -1, -1, +1, +1, -1, -1)$$

$$I \otimes I \otimes Z \qquad \mathrm{diag}(+1, -1, +1, -1, +1, -1, +1, -1)$$