Practical Quantum Computing

Lecture 03
Linear Algebra, Circuit Identities

with slides from Dave Bacon https://homes.cs.washington.edu/~dabacon/teaching/siena/

Week	**Tuesday** (3h)			**Wednesday** (3h)			**Deadlines**	
1. The Basics	Introduction	Gates	Circuit Identities	Qiskit	Cirq/Qualtran	Q&A		
	Programming Assignment 1: The basics of a quantum circuit simulator			**Programming Assignment 1:** The building blocks of a quantum circuit simulator				
2. Entanglement and its Applications	Teleportation	Superdense Coding	Quantum Key Distribution	PennyLane	Terminology of Projects	Q&A		
	Programming Assignment 2: The basics of a quantum circuit optimizer			**Programming Assignment 2:** The building blocks of a quantum circuit optimizer				
3. Computing	Phase Kickback and Toffoli	Distinguishing quantum states and The First Algorithms	Grover's Algorithm	Invited TBA		Q&A		11 May 2024
4. Advanced Topics*	Arithmetic Circuits*	Fault-Tolerance*	QML*	Invited TBA	Crumble	Q&A	18 May 2024	

* not evaluated

Learning goals - 03 Circuit Identities (The Basics)

1. What you have learned by now
   a. Quantum software: what, why and how
   b. Quantum circuits: diagrams and mathematics
2. **Mathematical notation for easier calculations**
   a. The Dirac bra-ket vectors
   b. Inner and outer products with the bra-ket notation
   c. Matrices as sums of outer products
3. **Computing with bra-kets**
   a. Unitary vectors and matrices
   b. Matrix vector multiplications using bra-kets
4. **Our first quantum circuit identities**
   a. Moving (commuting) single qubit gates through a circuit
   b. Expressing complicated (two-)qubit gates as products of elementary quantum gates

In the exercise session and programming assignment of this week

- basics of quantum circuit simulator
- build our own quantum circuit simulator

Dirac Notation: Bras and Kets

$$|v\rangle \quad \langle w|v\rangle \quad |\langle w|U|v\rangle|^2$$

$$|v\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle$$

$$|v\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$U_{QFT} = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \sum_{x=0}^{N-1} \omega_N^{xy} |y\rangle\langle x|$$

Dirac Notation: Bras and Kets

$$|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

"ket" := column vector

$$\langle w| = \begin{bmatrix} w_0 & w_1 & \cdots w_{N-1} \end{bmatrix}$$

"bra" := row vector

Every ket has a unique bra obtained by **complex conjugating** and **transposing**:

$$|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

$$(|v\rangle)^\dagger = \langle v| = \begin{bmatrix} v_0^* & v_1^* & \cdots v_{N-1}^* \end{bmatrix}$$

The Inner Product

Given a “bra” and a “ket” we can calculate an “inner product”

$$|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

$$\langle w| = \begin{bmatrix} w_0^* & w_1^* & \cdots w_{N-1}^* \end{bmatrix}$$

$$\langle w|v\rangle = \begin{bmatrix} w_0^* & w_1^* & \cdots w_{N-1}^* \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

$$= w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

This is a generalization of the dot product for real vectors

The result of taking an inner product is a complex number

The Inner Product in Comp. Basis

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

$$\langle w| = w_0^* \langle 0| + w_1^* \langle 1| + \cdots + w_{N-1}^* \langle N-1|$$

$$|v\rangle = v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle$$

$$\begin{aligned}\langle w|v\rangle \;=\; & (w_0^*\langle 0| + w_1^*\langle 1| + \cdots + w_{N-1}^*\langle N-1|) \\ & (v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle)\end{aligned}$$

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

Example: $|v\rangle = |0\rangle + 2i|1\rangle$ $\quad |w\rangle = 3i|0\rangle + (2i+2)|1\rangle$

$$\langle w|v\rangle = -3i \cdot 1 + (-2i+2)2i = 4 + i$$

Norm of a Vector

$$|||v\rangle|| = \sqrt{\langle v|v\rangle}$$

$$\langle v|v\rangle = v_0^* v_0 + v_1^* v_1 + \cdots + v_{N-1}^* v_{N-1}$$
$$= |v_0|^2 + |v_1|^2 + \cdots + |v_{N-1}|^2$$

which is always a positive real number

it is the length of the complex vector

Example: $|v\rangle = |0\rangle + 2i|1\rangle$

$$\langle v|v\rangle = |1|^2 + |2i|^2 = 5$$

$$|||v\rangle|| = \sqrt{5}$$

A Different Basis

A different orthonormal basis:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$\langle +|+\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1$$

$$\langle -|-\rangle = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{-1}{\sqrt{2}}\right|^2 = 1$$

$$\langle +|-\rangle = \left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\right) = 0$$

An orthonormal basis is complete if the number of basis elements is equal to the dimension of the complex vector space.

Changing Your Basis

Express the qubit wave function $|v\rangle = v_0|0\rangle + v_1|1\rangle$ in the orthonormal complete basis

$|a\rangle, |b\rangle$

in other words find components of $|v\rangle = v_a|a\rangle + v_b|b\rangle$

Some inner products:

$$\langle a|v\rangle = \langle a|\left(v_a|a\rangle + v_b|b\rangle\right) = v_a\langle a|a\rangle + v_b\langle a|b\rangle = v_a$$

$$\langle b|v\rangle = \langle b|\left(v_a|a\rangle + v_b|b\rangle\right) = v_a\langle b|a\rangle + v_b\langle b|b\rangle = v_b$$

So: $|v\rangle = (\langle a|v\rangle)|a\rangle + (\langle b|v\rangle)|b\rangle$

Calculating these inner products allows us to express the ket in a new basis.

Matrices, Bras, and Kets

We can expand a matrix about all of the computational basis outer products

$$M = \sum_{i,j=0}^{N-1} M_{i,j} |i\rangle\langle j| = \begin{bmatrix} M_{0,0} & \cdots & M_{N-1,0} \\ \vdots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

This makes it easy to operate on kets and bras:

$$M|v\rangle = \sum_{i,j=0}^{N-1} M_{i,j} |i\rangle\langle j|v\rangle$$

$$\langle j|k\rangle = \delta_{j,k} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ otherwise} \end{cases}$$

$$\langle w|M = \sum_{i,j=0}^{N-1} M_{i,j} \langle w|i\rangle\langle j|$$

Projectors

The projector onto a state $|v\rangle$ (which is of unit norm) is given by

$$P_v = |v\rangle\langle v|$$

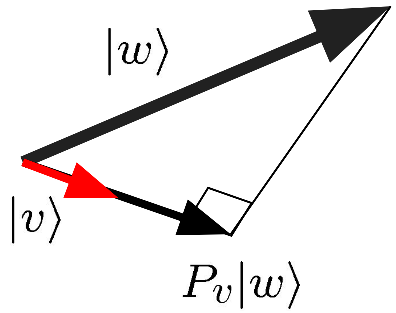
Note that

$$P_v|v\rangle = |v\rangle\langle v|v\rangle = |v\rangle$$

and that

$$P_v|w\rangle = |v\rangle\langle v|w\rangle = (\langle v|w\rangle)|v\rangle$$

Projects onto the state:



Example: $|v\rangle = |0\rangle$ $\qquad P_v = |0\rangle\langle 0|$

$$|w\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$P_v|w\rangle = |0\rangle\langle 0|\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) = \frac{1}{2}|0\rangle$$

Unitary Matrices

A matrix U is unitary if

$$U^\dagger U = I$$

N x N identity matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix} = diag(1, 1, \ldots, 1)$$

Equivalently a matrix U is unitary if

$$UU^\dagger = I$$

Unitary Evolution and the Norm

$$|v'\rangle = U|v\rangle$$

What happens to the norm $\langle v'|v'\rangle$ of the ket?

$$\langle v'| = (U|v\rangle)^\dagger = \langle v|U^\dagger$$

$$\langle v'|v'\rangle = \langle v|U^\dagger U|v\rangle = \langle v|I|v\rangle = \langle v|v\rangle$$

Unitary evolution does not change the length of the ket.

Normalized wave function

$$\sqrt{\langle v|v\rangle} = 1$$

$$|v'\rangle \xrightarrow{\quad} = U|v\rangle$$

unitary evolution

Normalized wave function

$$\sqrt{\langle v'|v'\rangle} = 1$$

This implies that unitary evolution will maintain being a unit vector

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Circuit Identities

$|0\rangle$ —[X]— $|1\rangle$

$|1\rangle$ —[X]— $|0\rangle$

"bit flip" is just the classical not gate

Hadamard gate:

$$-\boxed{H}- = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

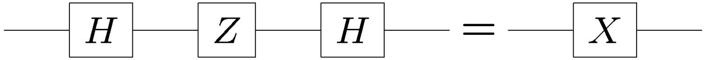
$$H^2 = I$$

$$-\boxed{H}-\boxed{Z}-\boxed{H}- = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

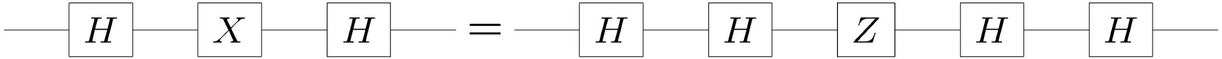
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\boxed{X}-$$

$$-\boxed{H}-\boxed{Z}-\boxed{H}- = -\boxed{X}-$$

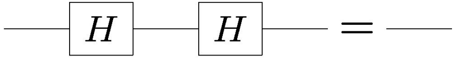
Circuit Identities



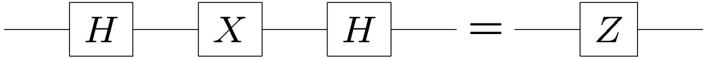
Use this to compute HXH



But $H^2 = I$

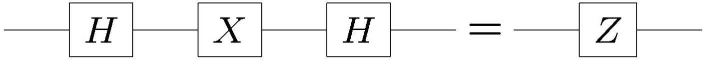


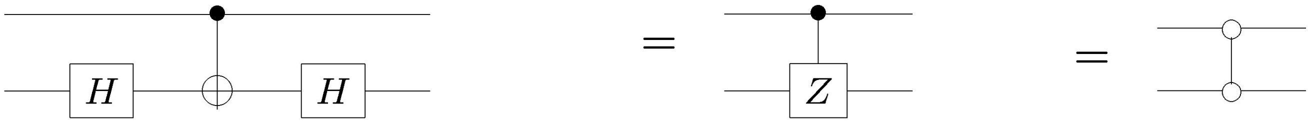
So that

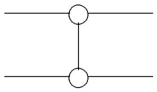


Circuit Identities

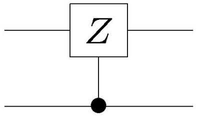
Using

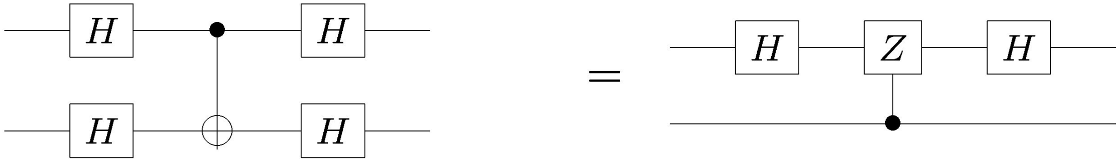


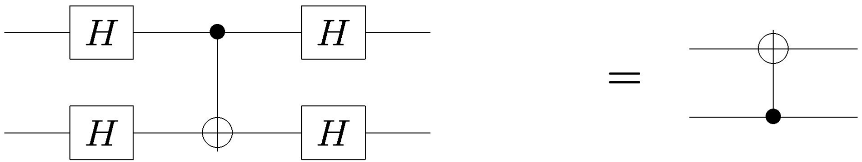




$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} =$$







Appendix

The Inner Product

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

Example:

$$|w\rangle = \begin{bmatrix} 3i \\ 3 \end{bmatrix}$$

$$|v\rangle = \begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$$

$$\langle w| = \begin{bmatrix} -3i & 3 \end{bmatrix}$$

$$\langle w|v\rangle = \begin{bmatrix} -3i & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1+2i \end{bmatrix} = (-3i)\cdot 1 + 3(1+2i) = 3+3i$$

$$\langle v|w\rangle = \begin{bmatrix} 1 & 1-2i \end{bmatrix} \begin{bmatrix} 3i \\ 3 \end{bmatrix} = 1\cdot(3i) + (1-2i)3 = 3-3i$$

Complex conjugate of inner product: $(\langle w|v\rangle)^* = \langle v|w\rangle$

The Inner Product in Comp. Basis

$$\langle w|v\rangle = w_0^* v_0 + w_1^* v_1 + \cdots + w_{N-1}^* v_{N-1}$$

$$\langle 0|0\rangle = \begin{bmatrix} 1 & 0 & \cdots 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1\cdot1 + 0\cdot0 + \cdots + 0\cdot0 = 1$$

$$\langle 0|1\rangle = \begin{bmatrix} 1 & 0 & \cdots 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = 1\cdot0 + 1\cdot0 + \cdots + 0\cdot0 = 0$$

Inner product of computational basis elements:

$$\langle j|k\rangle = \delta_{j,k} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ otherwise} \end{cases}$$

Kronecker delta

Computational Basis

Some special vectors:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \qquad \bullet\bullet\bullet \qquad |N-1\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example:

2 dimensional complex vectors (also known as: a qubit!)

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Computational Basis

Vectors can be “expanded” in the computational basis:

$$|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} = v_0 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_1 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + v_{N-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= v_0|0\rangle + v_1|1\rangle + \cdots + v_{N-1}|N-1\rangle$$

Example:

$$|v\rangle = \begin{bmatrix} 1+2i \\ 3 \end{bmatrix} = (1+2i)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= (1+2i)|0\rangle + 3|1\rangle$$

Computational Basis

Computational Basis, but now for bras:

$$\langle 0| = \begin{bmatrix} 1 & 0 & \cdots 0 \end{bmatrix}$$

$$\langle 1| = \begin{bmatrix} 0 & 1 & \cdots 0 \end{bmatrix}$$

$$\vdots$$

$$\langle N-1| = \begin{bmatrix} 0 & 0 & \cdots 1 \end{bmatrix}$$

$$\langle v| = \begin{bmatrix} v_0^* & v_1^* & \cdots v_{N-1}^* \end{bmatrix} = v_0^*\langle 0| + v_1^*\langle 1| + \cdots + v_{N-1}^*\langle N-1|$$

Example:

$$\langle v| = \begin{bmatrix} 2 & 3+2i \end{bmatrix} = 2\langle 0| + (3+2i)\langle 1|$$

Computational Basis

Computational basis:

$$|0\rangle$$

$$|1\rangle$$

$$\vdots$$

$$|N-1\rangle$$

is an orthonormal basis:

$$\langle j|k\rangle = \delta_{j,k} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ otherwise} \end{cases}$$

Kronecker delta

Computational basis is important because when we measure our quantum computer (a qubit, two qubits, etc.) we get an outcome corresponding to these basis vectors.

But there are all sorts of other basis which we could use to, say, expand our vector about.

Complex Vectors, Addition

Complex vectors can be added

$$|v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix} \qquad |w\rangle = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{bmatrix}$$

$$|v\rangle + |w\rangle = \begin{bmatrix} v_0 + w_0 \\ v_1 + w_1 \\ \vdots \\ v_{N-1} + w_{N-1} \end{bmatrix}$$

Addition and multiplication by a scalar:

$$\alpha|v\rangle + \beta|w\rangle = \begin{bmatrix} \alpha v_0 + \beta w_0 \\ \alpha v_1 + \beta w_1 \\ \vdots \\ \alpha v_{N-1} + \beta w_{N-1} \end{bmatrix}$$

Explicit Basis Change

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express $|v\rangle = |0\rangle$ in this basis:

$$|v\rangle = v_+|+\rangle + v_-|-\rangle$$

$$\langle +|0\rangle = \frac{1}{\sqrt{2}}$$

$$|v\rangle = (\langle +|v\rangle)|+\rangle + (\langle -|v\rangle)|-\rangle$$

$$\langle -|0\rangle = \frac{1}{\sqrt{2}}$$

So:

$$|v\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} \end{bmatrix}$$

Example Basis Change

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express $|v\rangle = v_0|0\rangle + v_1|1\rangle$ in this basis: $|v\rangle = v_+|+\rangle + v_-|-\rangle$

$$\langle +|v\rangle = \langle +|\left(v_+|+\rangle + v_-|-\rangle\right) = v_+\langle +|+\rangle + v_-\langle +|-\rangle = v_+$$

$$\langle -|v\rangle = \langle -|\left(v_+|+\rangle + v_-|-\rangle\right) = v_+\langle -|+\rangle + v_-\langle -|-\rangle = v_-$$

So: $|v\rangle = (\langle +|v\rangle)|+\rangle + (\langle -|v\rangle)|-\rangle$

$$\langle +|v\rangle = \left(\frac{1}{\sqrt{2}}v_0\right) + \left(\frac{1}{\sqrt{2}}v_1\right) = \frac{v_0 + v_1}{\sqrt{2}}$$

$$\langle -|v\rangle = \left(\frac{1}{\sqrt{2}}v_0\right) + \left(\frac{-1}{\sqrt{2}}v_1\right) = \frac{v_0 - v_1}{\sqrt{2}}$$

$$|v\rangle = \frac{v_0 + v_1}{\sqrt{2}}|+\rangle + \frac{v_0 - v_1}{\sqrt{2}}|-\rangle$$

Matrices

A N dimensional complex matrix M is an N by N array of complex numbers:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$M_{j,k}$ are complex numbers

Example:

Three dimensional complex matrix:

$$M = \begin{bmatrix} 4 & 3+i & 2 \\ i & e^{\frac{\pi}{4}} & \sqrt{2}i \\ 0 & 0 & 4 \end{bmatrix}$$

$$M_{1,0} = i$$

$$M_{2,2} = 4$$

Matrices, Bras, and Kets

We can expand a matrix about all of the **computational basis outer products**

$$M = \sum_{i,j=0}^{N-1} M_{i,j} |i\rangle\langle j| = \begin{bmatrix} M_{0,0} & \cdots & M_{N-1,0} \\ \vdots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$$

$$|0\rangle\langle 0| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad |1\rangle\langle 1| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$M = |0\rangle\langle 0| + i|0\rangle\langle 1| - 1|1\rangle\langle 0| - i|1\rangle\langle 1|$$

Matrices, Added

Matrices can be added

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} \qquad L = \begin{bmatrix} L_{0,0} & \cdots & L_{0,N-1} \\ \vdots & & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} \end{bmatrix}$$

$$M+L = \begin{bmatrix} M_{0,0} + L_{0,0} & \cdots & M_{0,N-1} + L_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} + L_{N-1,0} & \cdots & M_{N-1,N-1} + L_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 3 & -3-i \\ i & 2 \end{bmatrix}$$

$$M+L = \begin{bmatrix} 0+3 & 3+i+(-3-i) \\ i+i & 1+2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2i & 3 \end{bmatrix}$$

Matrices, Complex Conjugate

Given a matrix, we can form its complex conjugate by conjugating every element:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^* = \begin{bmatrix} M^*_{0,0} & \cdots & M^*_{0,N-1} \\ \vdots & & \vdots \\ M^*_{N-1,0} & \cdots & M^*_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix} \qquad M^* = \begin{bmatrix} 0 & 3-i \\ -i & 1 \end{bmatrix}$$

Matrices, Transpose

Given a matrix, we can form it's transpose by reflecting across the diagonal

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^T = \begin{bmatrix} M_{0,0} & \cdots & M_{N-1,0} \\ \vdots & & \vdots \\ M_{0,N-1} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix} \qquad M^T = \begin{bmatrix} 0 & i \\ 3+i & 1 \end{bmatrix}$$

Matrices, Conjugate Transpose

Given a matrix, we can form its conjugate transpose by reflecting across the diagonal and conjugating

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$M^\dagger = \begin{bmatrix} M^*_{0,0} & \cdots & M^*_{N-1,0} \\ \vdots & & \vdots \\ M^*_{0,N-1} & \cdots & M^*_{N-1,N-1} \end{bmatrix}$$

Example:

$$M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix} \qquad M^\dagger = \begin{bmatrix} 0 & -i \\ 3-i & 1 \end{bmatrix}$$

Matrices, Multiplied by Scalar

Matrices can be multiplied by a complex number

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$\alpha M = \begin{bmatrix} \alpha M_{0,0} & \cdots & \alpha M_{0,N-1} \\ \vdots & & \vdots \\ \alpha M_{N-1,0} & \cdots & \alpha M_{N-1,N-1} \end{bmatrix}$$

Example: $M = \begin{bmatrix} 0 & 3+i \\ i & 1 \end{bmatrix}$ $\alpha = 2i$

$$\alpha M = \begin{bmatrix} 2i \cdot 0 & 2i(3+i) \\ 2i(i) & 2i(1) \end{bmatrix} = \begin{bmatrix} 0 & -2+6i \\ -2 & 2i \end{bmatrix}$$

Matrices, Multiplied

Matrices can be multiplied

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} \qquad L = \begin{bmatrix} L_{0,0} & \cdots & L_{0,N-1} \\ \vdots & & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} \end{bmatrix}$$

$$R = ML = \begin{bmatrix} R_{0,0} & \cdots & R_{0,N-1} \\ \vdots & & \vdots \\ R_{N-1,0} & \cdots & R_{N-1,N-1} \end{bmatrix}$$

$$R_{0,0} = M_{0,0}L_{0,0} + M_{0,1}L_{1,0} + \cdots + M_{0,N-1}L_{N-1,0}$$

$$R_{0,1} = M_{0,0}L_{0,1} + M_{0,1}L_{1,1} + \cdots + M_{0,N-1}L_{N-1,1}$$

$$\vdots$$

$$R_{j,k} = M_{j,0}L_{0,k} + M_{j,1}L_{1,k} + \cdots + M_{j,N-1}L_{N-1,k}$$

$$R_{j,k} = \sum_{l=0}^{N-1} M_{j,l}L_{l,k}$$

Matrices and Kets, Multiplied

Given a matrix, and a column vector:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} \qquad |v\rangle = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

These can be multiplied to obtain a new column vector:

$$M|v\rangle = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} \begin{bmatrix} v_0 \\ \vdots \\ v_{N-1} \end{bmatrix}$$

$$= \begin{bmatrix} M_{0,0}v_0 + M_{0,1}v_1 + \cdots M_{0,N-1}v_{N-1} \\ \vdots \\ M_{N-1,0}v_0 + M_{N-1,1}v_1 + \cdots M_{N-1,N-1}v_{N-1} \end{bmatrix}$$

Matrices and Bras, Multiplied

Given a matrix, and a row vector:

$$M = \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix} \qquad \langle w| = \begin{bmatrix} w_0^* & w_1^* & \cdots w_{N-1}^* \end{bmatrix}$$

These can be multiplied to obtain a new row vector:

$$\langle w|M = \begin{bmatrix} w_0^* & \cdots & w_{N-1}^* \end{bmatrix} \begin{bmatrix} M_{0,0} & \cdots & M_{0,N-1} \\ \vdots & & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-1} \end{bmatrix}$$

$$\langle w|M = \begin{bmatrix} r_0^* & \cdots & r_{N-1}^* \end{bmatrix}$$

$$r_0^* = w_0^* M_{0,0} + \cdots + w_{N-1}^* M_{N-1,0}$$

$$\vdots$$

$$r_{N-1}^* = w_0^* M_{0,N-1} + \cdots + w_{N-1}^* M_{N-1,N-1}$$