$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

$$A\underline{x} = \underline{b}$$

FOR EXAMPLE

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n}$

$\underline{x} \in \mathbb{R}^n$

$\underline{b} \in \mathbb{R}^m$

Problem

Given A and $\underline{b}$,
find $\underline{x}$

Def

The LINEAR SPAN of $q_1, q_2, \dots, q_k \in \mathbb{R}^n$ is the set

$$\left\{ \underline{z} \in \mathbb{R}^n \text{ s.t. } \underline{z} = \sum_{i=1}^{k} \alpha_i q_i \text{ for some } \alpha_1, \dots, \alpha_k \in \mathbb{R} \right\}$$

and is denoted by $\mathrm{span}(q_1, \dots, q_k)$

$$q_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad q_2 := \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\alpha_1 = 1 \qquad \alpha_2 = 42$$

$$z = 1 \cdot q_1 + 42 \cdot q_2 = \begin{bmatrix} 1 \\ 127 \end{bmatrix}$$

Def

The vector $q_1, \dots, q_k \in \mathbb{R}^n$ are LINEARLY INDEPENDENT

if $\sum_{i=1}^{k} \alpha_i q_i = \underline{0} \iff \alpha_1 = \dots = \alpha_k = 0$

(Otherwise, they are LINEARLY DEPENDENT)

$$q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\alpha_1 = -2 \qquad \alpha_2 = 1$$

$$-2\, q_1 + q_2 = \underline{0}$$

$\Rightarrow$ LIN. DEPENDENT

$$q_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\alpha_1 q_1 + \alpha_2 q_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \underline{0}$$

If $q_1, \dots, q_k \in \mathbb{R}^n$, I can define

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_k \end{bmatrix} \in \mathbb{R}^{n \times k}$$

$$\mathrm{span}(q_1, \dots, q_k) = \left\{ \underline{z} \in \mathbb{R}^n \text{ s.t. } z = Q\underline{\alpha} \text{ for some } \underline{\alpha} \in \mathbb{R}^k \right\}$$

The columns of the matrix Q are lin. independent

$$\underline{Q}\,\underline{\alpha} = \underline{0} \iff \underline{\alpha} = \underline{0}$$

Def

Let $A \in \mathbb{R}^{m\times n}$. Then the RANGE of A is

$$R(A) = \{\underline{y} \in \mathbb{R}^m \text{ s.t. } \underline{y} = A\underline{x} \text{ for some } \underline{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

and the NULL SPACE of A is

$$N(A) = \{\underline{z} \in \mathbb{R}^n \text{ s.t. } A\underline{z} = \underline{0}\} \subseteq \mathbb{R}^n$$

Suppose $\underline{x}, \underline{y} \in N(A)$. Then $\alpha\underline{x} \in N(A)$ $(\forall \alpha \in \mathbb{R})$ and $\underline{x}+\underline{y} \in N(A)$

$$A(\alpha\underline{x}) = \alpha(A\underline{x}) = \alpha\underline{0} = \underline{0} \Longrightarrow \alpha\underline{x} \in N(A)$$

$$A(\underline{x}+\underline{y}) = A\underline{x} + A\underline{y} = \underline{0}+\underline{0} = \underline{0} \Rightarrow \underline{x}+\underline{y} \in N(A)$$

If $\underline{x}, \underline{y} \in R(A)$ then $\alpha\underline{x}+\beta\underline{y} \in R(A)$

$\underline{x} \in R(A)$ then there exists $\underline{z}$ s.t. $\underline{x} = A\underline{z}$

$\underline{y} \in R(A) \Longrightarrow \exists\, \underline{w}$ s.t. $\underline{y} = A\underline{w}$

$$A(\alpha\underline{z}+\beta\underline{w}) = A(\alpha\underline{z})+A(\beta\underline{w}) = \alpha(A\underline{z})+\beta(A\underline{w}) =$$

$$= \alpha\underline{x}+\beta\underline{y} \Longrightarrow \alpha\underline{x}+\beta\underline{y} \in R(A)$$

Def

Let $E \subseteq \mathbb{R}^n$ be non-empty and s.t. for all $\underline{x}, \underline{y} \in E$ and $\alpha \in \mathbb{R}$ then

(1) $\underline{x} + \underline{y} \in E$ (2) $\alpha \underline{x} \in E$.

E is called a SUBSPACE of $\mathbb{R}^n$

$\{\underline{0}\} \subset \mathbb{R}^n$ is the TRIVIAL SUBSPACE

$$A\underline{x} = \underline{b} \qquad (1)$$

A solution $\underline{x}$ exists if and only if $\underline{b} \in R(A)$

LEMMA

Suppose that (1) has a solution. Then, it is unique if and only if $N(A) = \{\underline{0}\}$.

Proof

① Suppose $N(A) \neq \{\underline{0}\}$. Then there is $\underline{0} \neq \underline{z}$ s.t. $A\underline{z} = \underline{0}$.

If $A\underline{x} = \underline{b}$ then $A(\underline{x} + \underline{z}) = A\underline{x} + A\underline{z} = \underline{b} + \underline{0} = \underline{b}$

② Suppose $N(A) = \{\underline{0}\}$ but

$$A\underline{x}_1 = \underline{b} = A\underline{x}_2, \quad \underline{x}_1 \neq \underline{x}_2$$

$$A(\underline{x}_1 - \underline{x}_2) = A\underline{x}_1 - A\underline{x}_2 = \underline{b} - \underline{b} = \underline{0}$$

$\Rightarrow \underline{0} = \underline{x}_1 - \underline{x}_2 \in N(A)$, contradiction.

Def

Let $E \subset \mathbb{R}^n$ be a subspace. The set

$$\{q_1, \dots, q_k\} \subset E \quad \text{are a BASIS of } E \text{ if}$$

1. $q_1, \dots, q_k$ are LINEARLY INDEPENDENT
2. $E = \mathrm{span}(q_1, \dots, q_k)$

$$\dim\{\underline{0}\} = 0$$

If $E \neq \{\underline{0}\}$, then $\dim E =$

\# vectors in a basis of E