$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$$

$$R(Q) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$= \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$R(Q) = \left\{ \underline{z} \text{ s.t. } \underline{z} = Q\underline{\alpha} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \\ 0 \end{bmatrix} = (\alpha_1 + \alpha_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \dim R(Q) = 2$$

$$N(Q) = \left\{ \underline{x} \text{ s.t. } Q\underline{x} = \underline{0} \right\} = \left\{ \underline{0} \right\} \Rightarrow \dim N(Q) = 0$$

THM

Let $A \in \mathbb{R}^{m \times n}$. Then, $\dim R(A) + \dim N(A) = n$.

Proof

Let $\dim N(A) = k$. Let $\{\underline{v}_1, \dots, \underline{v}_k\}$ be a basis of $N(A)$.

and let $\{\underline{v}_1, \dots, \underline{v}_k, \underline{w}_1, \dots, \underline{w}_{n-k}\}$ be a basis of $\mathbb{R}^n$.

$V = [\underline{v}_1, \dots, \underline{v}_k] \in \mathbb{R}^{n \times k}$ $\quad W = [\underline{w}_1, \dots, \underline{w}_{n-k}] \in \mathbb{R}^{n \times (n-k)}$

$[V \ W]$ is a basis of $\mathbb{R}^n \Rightarrow \underline{x} \in \mathbb{R}^n$ can be written $\underline{x} = [V \ W]\underline{c}$

$R(A) = \{ y \in \mathbb{R}^m \text{ s.t. } y = A\underline{x} = A[V \ W]\underline{c} \text{ for some } \underline{c} \in \mathbb{R}^n \}$

$$A[V \ W] = [A\underline{v}_1, \dots, A\underline{v}_k, A\underline{w}_1, \dots, A\underline{w}_{n-k}] = [\underline{0}, \dots, \underline{0}, A\underline{w}_1, \dots, A\underline{w}_{n-k}]$$

$$A\begin{bmatrix} V & W \end{bmatrix}\underline{c} = \begin{bmatrix} \underline{0} & \cdots & \underline{0} & A\underline{w}_1 & \cdots & A\underline{w}_{n-k} \end{bmatrix}\underline{c} = \sum_{i=1}^{n-k} (A\underline{w}_i) c_{i+k}$$

$$R(A) = \operatorname{span}(A\underline{w}_1, \ldots, A\underline{w}_{n-k})$$

$$AW\underline{\alpha} = \underline{0} \Leftarrow \underline{\alpha} = \underline{0} \Leftarrow \begin{bmatrix} \underline{\beta} \\ -\underline{\alpha} \end{bmatrix} = \underline{0}$$

$$\Updownarrow \qquad\qquad\qquad \Uparrow$$

$$W\underline{\alpha} \in N(A)$$

$$\Updownarrow$$

$$W\underline{\alpha} = V\underline{\beta} \iff \begin{bmatrix} V & W \end{bmatrix}\begin{bmatrix} \underline{\beta} \\ -\underline{\alpha} \end{bmatrix} = \underline{0}$$

$\therefore \; \{A\underline{w}_1, \dots, A\underline{w}_{n-k}\}$ are a basis of $R(A)$

Hence, $\dim N(A) = k$ and $\dim R(A) = n-k$

So $\dim N(A) + \dim R(A) = n$.

E subspace with basis $\{q_1, \dots, q_k\}$ $\quad Q = [q_1, \dots, q_k] \in \mathbb{R}^{n \times k}$

Then, if $\underline{x} \in E$ then $\underline{x} = \sum \alpha_i q_i$

Moreover, $\underline{\alpha} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix}$ is unique. Indeed, suppose $\underline{x} = Q\underline{\alpha} = Q\underline{\beta}$.

Then, $Q(\underline{\alpha} - \underline{\beta}) = \underline{0} \Rightarrow \underline{\alpha} - \underline{\beta} = \underline{0} \Rightarrow \underline{\alpha} = \underline{\beta}$

$A \in \mathbb{R}^{m \times n}$

$A\underline{x} = \underline{b}$ When $\exists!\ \underline{x}$ for all $\underline{b}$? (Q1)

For existence, we need $R(A) = \mathbb{R}^m$

For uniqueness, we need $N(A) = \{\underline{0}\}$

COROLLARY

Let $A \in \mathbb{R}^{n\times n}$. Then, the answer to (Q1) is "yes" iff any of the following equivalent conditions holds:

① $R(A) = \mathbb{R}^n$

② The columns of A are lin. independent

③ $N(A) = \{0\}$

④ $\det A \neq 0$

⑤ 0 is not an eigenvalue of A

$$A\underline{x} = \underline{b} \qquad A\underline{y} = \underline{c}$$

$$A(\alpha\underline{x} + \beta\underline{y}) = \alpha\underline{b} + \beta\underline{c}$$

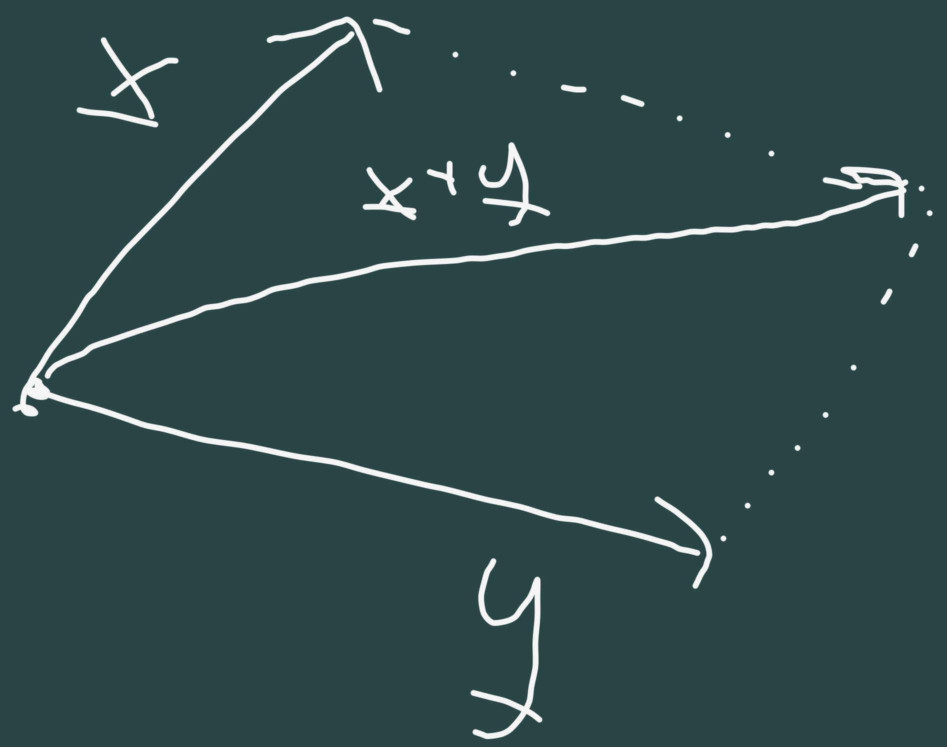
$$A^{-1}\underline{b} = \underline{x} \qquad AA^{-1}\underline{b} = A\underline{x} = \underline{b} = I \cdot \underline{b}$$

DEF

A function $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a NORM if:

1. $\|\underline{x}\| \geq 0$ for all $\underline{x} \in \mathbb{R}^n$ and $\|\underline{x}\| = 0 \Leftrightarrow \underline{x} = \underline{0}$
2. $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$ for all $\underline{x}, \underline{y} \in \mathbb{R}^n$ (TRIANGLE INEQUALITY)
3. $\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\|$ for all $\underline{x} \in \mathbb{R}^n$, for all $\alpha \in \mathbb{R}$

In $\mathbb{R}^2$, the Euclidean length: $\left\|\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right\| \longmapsto \sqrt{x_1^2+x_2^2}$



$$\left\|\begin{bmatrix}\alpha x_1\\ \alpha x_2\end{bmatrix}\right\| = \sqrt{(\alpha x_1)^2+(\alpha x_2)^2} =$$

$$= \sqrt{\alpha^2 x_1^2+\alpha^2 x_2^2} = \sqrt{\alpha^2(x_1^2+x_2^2)} = |\alpha|\sqrt{x_1^2+x_2^2}$$

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$ is a norm for all $1 \leq p < \infty$

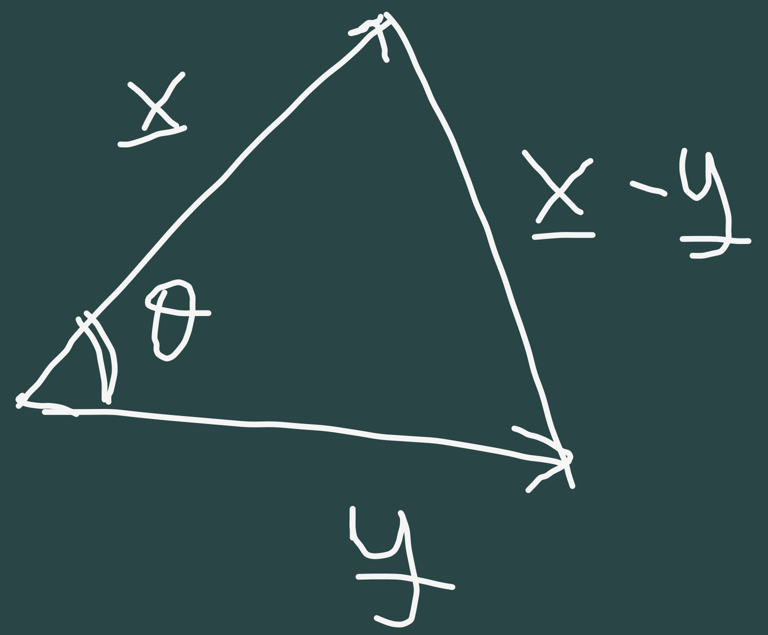
$$\|\underline{x}\|_\infty = \max_i |x_i|$$ is a norm.

$$A\underline{x} = \underline{b}$$

$$A\underline{y} = \underline{b} + \underline{\delta b}$$

$$\|\underline{\delta b}\| < 10^{-200}$$

$$\|\underline{y} - \underline{x}\|$$



$$\|\underline{x}\|_2^2 + \|\underline{y}\|_2^2 - 2\|\underline{x}\|_2\|\underline{y}\|_2\cos\theta = \|\underline{x}-\underline{y}\|_2^2$$

If $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \|\underline{x}-\underline{y}\|_2^2 = (x_1-y_1)^2 + (x_2-y_2)^2$

$$= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2(x_1 y_1 + x_2 y_2)$$

$$\cos\theta = \frac{x_1 y_1 + x_2 y_2}{\|\underline{x}\|_2 \|\underline{y}\|_2}$$

$$\underline{x}\cdot\underline{y} = \sum_{i=1}^{n} x_i y_i = \underline{x}^T \underline{y}$$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

DEF

A function $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is an INNER PRODUCT if

1. $\langle \underline{x}, \underline{x} \rangle \geq 0$ for all $\underline{x} \in \mathbb{R}^n$ and $\langle \underline{x}, \underline{x} \rangle = 0 \Leftrightarrow \underline{x} = \underline{0}$
2. $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$ for all $\underline{x}, \underline{y} \in \mathbb{R}^n$
3. $\langle \alpha \underline{x} + \beta \underline{y}, \underline{z} \rangle = \alpha \langle \underline{x}, \underline{z} \rangle + \beta \langle \underline{y}, \underline{z} \rangle$ for all $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^n$ and for all $\alpha, \beta \in \mathbb{R}$