Practical Quantum Computing

Lecture 04
Teleportation

with slides from Dave Bacon https://homes.cs.washington.edu/~dabacon/teaching/siena/

Week	**Tuesday** (3h)			**Wednesday** (3h)			**Deadlines**	
1. The Basics	Introduction	Gates	Circuit Identities	Qiskit	Cirq/Qualtran	Q&A		
	Programming Assignment 1: The basics of a quantum circuit simulator			**Programming Assignment 1:** The building blocks of a quantum circuit simulator				
2. Entanglement and its Applications	Teleportation	Superdense Coding	Quantum Key Distribution	Qualtran/Assignment2	Terminology of Projects	Q&A		
	Programming Assignment 2: The basics of a quantum circuit optimizer			**Programming Assignment 2:** The building blocks of a quantum circuit optimizer				
3. Computing	Phase Kickback and Toffoli	Distinguishing quantum states and The First Algorithms	Grover's Algorithm	Invited TBA	PennyLane	Q&A		11 May 2024
4. Advanced Topics*	Arithmetic Circuits*	Fault-Tolerance*	QML*	Invited TBA	Crumble	Q&A	18 May 2024	

* not evaluated

Quiz/Discussion from week 1

What are (dynamic) quantum circuits?

Learning goals - 04 Teleportation (Entanglement)

1. What you have learned by now
   a. Quantum software: what, why and how
   b. Quantum circuits: mathematics, diagrams and circuit identities
2. **Quantum Information cannot be copied**
   a. Measuring quantum bits and the visible information
   b. No-Cloning Theorem
3. **Quantum information can be teleported**
   a. What is teleportation?
   b. How can teleportation be derived by using circuit identities?
4. **Entanglement (without theory) for teleportation**
   a. *Special* two qubit states - Definition of Bell states
   b. Understanding the quantum measurement of Bell states
   c. Implementing teleportation using Bell states

In the exercise session and programming assignment of this week

- basics of quantum circuit optimization
- build our own quantum circuit optimizer
- benchmark your optimizer

Measurement Rule

If we measure a quantum system whose $|v\rangle$ wave function is in the basis $|w_i\rangle$, then the probability of getting the outcome corresponding to $|w_i\rangle$ is given by

$$Pr(|w_i\rangle) = |\langle w_i|v\rangle|^2 = \langle v|w_i\rangle\langle w_i|v\rangle = \langle v|P_{w_i}|v\rangle$$

where

$$P_{w_i} = |w_i\rangle\langle w_i|$$

The new wave function of the system after getting the measurement outcome corresponding to $|w_i\rangle$ is given by

$$|v'\rangle = \frac{P_{w_i}|v\rangle}{\sqrt{Pr(|w_i\rangle)}}$$

Measuring One of Two Qubits

Suppose we measure the first of two qubits in the computational basis. Then we can form the two projectors:

$$P_0 \otimes I = |0\rangle\langle 0| \otimes I$$
$$P_1 \otimes I = |1\rangle\langle 1| \otimes I$$

$$I = |0\rangle\langle 0| + |1\rangle\langle 1|$$

If the two qubit wave function is $|v\rangle$ then the probabilities of these two outcomes are

$$Pr(0) = \langle v|P_0 \otimes I|v\rangle$$
$$Pr(1) = \langle v|P_1 \otimes I|v\rangle$$

And the new state of the system is given by either

$$|v'\rangle = \frac{P_0 \otimes I|v\rangle}{\sqrt{Pr(0)}} \qquad |v'\rangle = \frac{P_1 \otimes I|v\rangle}{\sqrt{Pr(1)}}$$

Instantaneous Communication?

Suppose two distant parties each have a qubit and their joint quantum wave function is

$$|v\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

If one party now measures its qubit, then…

$$P_0 \otimes I = |0\rangle\langle 0| \otimes I \qquad Pr(0) = \frac{1}{2} \qquad |v'\rangle = |0\rangle \otimes |0\rangle$$

$$P_1 \otimes I = |1\rangle\langle 1| \otimes I \qquad Pr(1) = \frac{1}{2} \qquad |v'\rangle = |1\rangle \otimes |1\rangle$$

The other parties qubit is now either the $|0\rangle$ or $|1\rangle$

Instantaneous communication? NO! These two results happen with probabilities.

No-Cloning Theorem

One could, with high probability, **learn** a qubit state, if there was a way to make copies of a qubit

- prior to making the measurement
- without running the whole computation over again

Copy the state of |q> to the clean register |0>

$$U|q>|0> = |q>|q>$$

Assume two arbitrary states |a> and |b>

$$U(|a>|0>)=|a>|a> \text{ and } U(|b>|0>)=|b>|b>$$

Moreover

$$U(a|a> + b|b>)|0> = U(a|a>|0>) + U(b|b>|0>) = a|aa> + b|bb>$$

$$U(a|a> + b|b>)|0> = (a|a> + b|b>)(a|a> + b|b>) = a^2|aa> + b^2|bb> + ab|ab> + ab|ba>$$

No-Cloning Theorem

Moreover

$$U(a|a> + b|b>)|0> = U(a|a>|0>) + U(b|b>|0>) = a|aa> + b|bb>$$

$$U(a|a> + b|b>)|0> = (a|a> + b|b>)(a|a> + b|b>) = a^2|aa> + b^2|bb> + ab|ab> + ab|ba>$$

Such that

$$a = a^2$$

$$b = b^2$$

$$ab = 0$$

Only if a=0 and b=0

Arbitrary quantum states cannot be copied !

Learning goals - 04 Teleportation (Entanglement)

1. What you have learned by now
   a. Quantum software: what, why and how
   b. Quantum circuits: mathematics, diagrams and circuit identities
2. **Quantum Information cannot be copied**
   a. Measuring quantum bits and the visible information
   b. No-Cloning Theorem
3. **Quantum information can be teleported**
   a. What is teleportation?
   b. How can teleportation be derived by using circuit identities?
4. **Entanglement (without theory) for teleportation**
   a. *Special* two qubit states - Definition of Bell states
   b. Understanding the quantum measurement of Bell states
   c. Implementing teleportation using Bell states

In the exercise session and programming assignment of this week

- basics of quantum circuit optimization
- build our own quantum circuit optimizer
- benchmark your optimizer

Quantum Teleportation

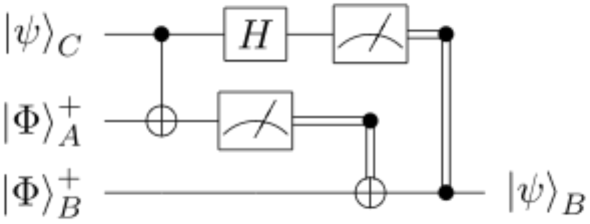
Alice wants to send her qubit to Bob.

She does not know the wave function of her qubit.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

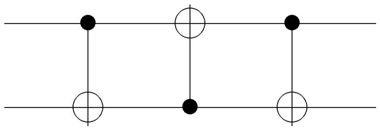
Can Alice send her qubit to Bob using classical bits?

Since she doesn't know $|\psi\rangle$ and measurements on her state do not reveal $|\psi\rangle$, this task appears impossible.

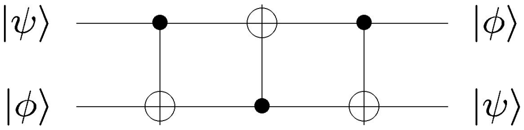


Deriving Quantum Teleportation

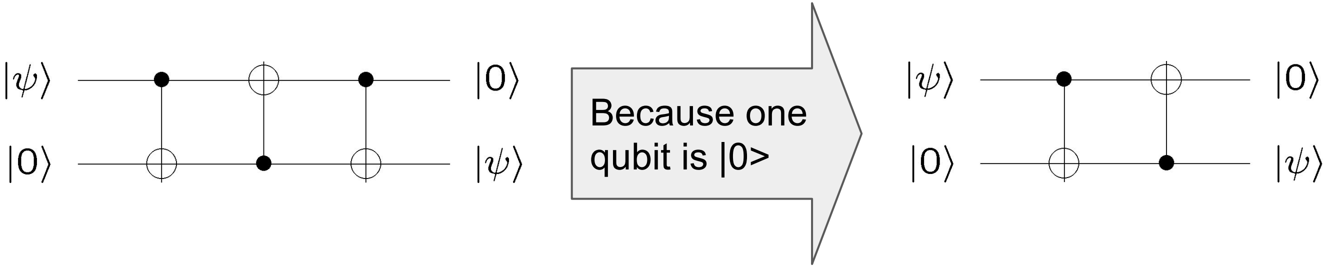
Our path: We are going to “derive” teleportation



$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

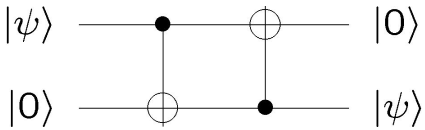


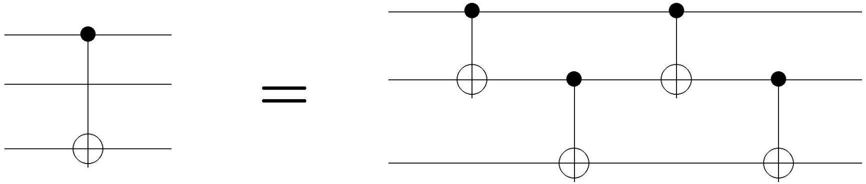
Only concerned with from Alice to Bob transfer

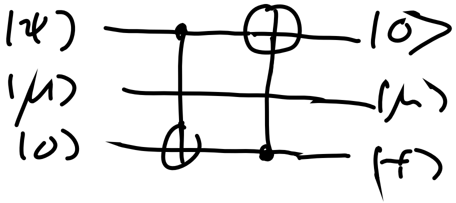


Deriving Quantum Teleportation

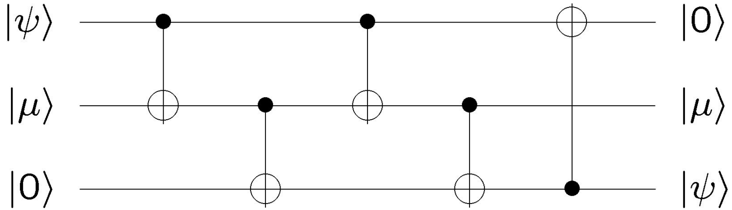
Need some way to get entangled states







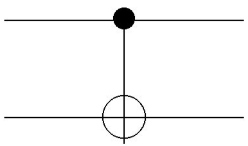
new equivalent circuit:



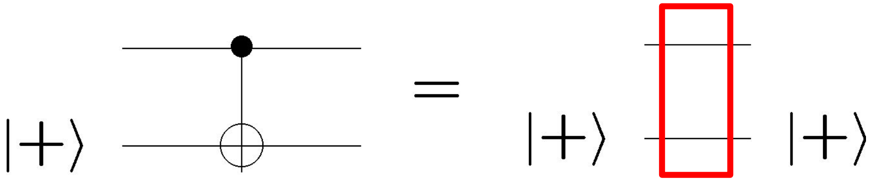
Deriving Quantum Teleportation

How to generate an entangled state:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

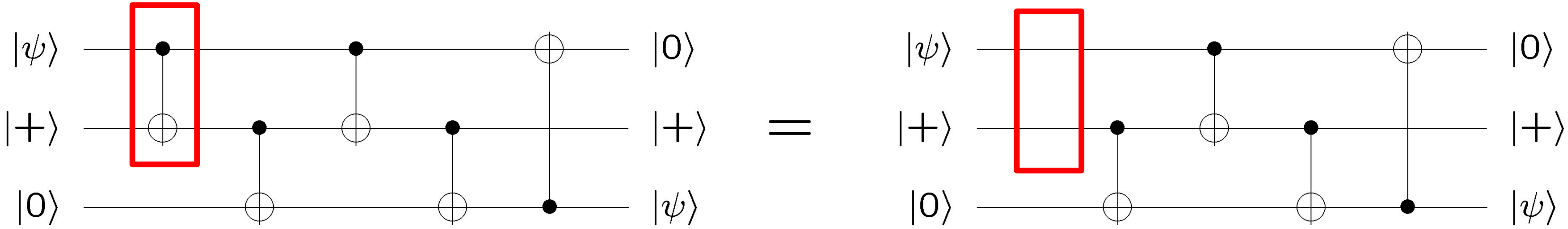


$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

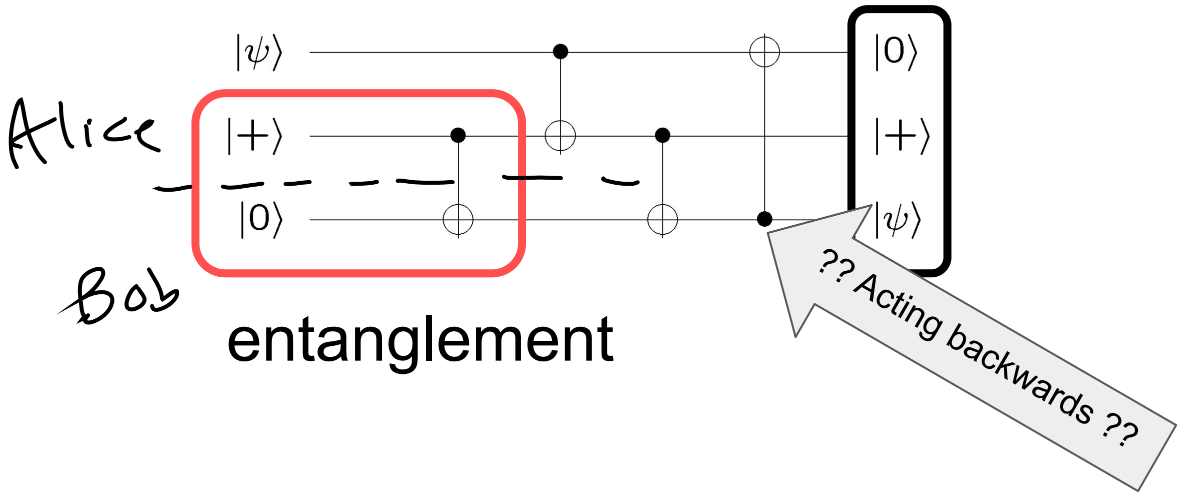


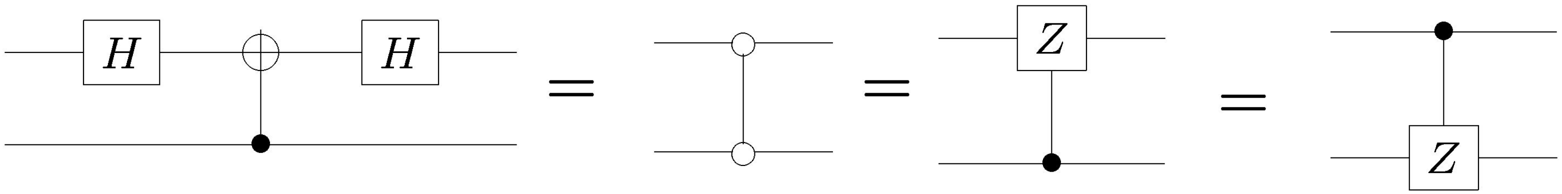
$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$X|+\rangle = |+\rangle$$



Deriving Quantum Teleportation





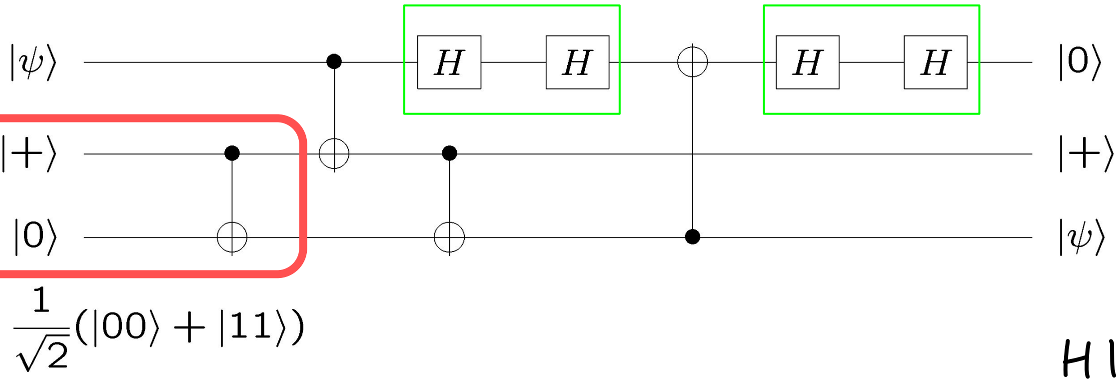
$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$HXH = Z$$

$$H^2 = I$$

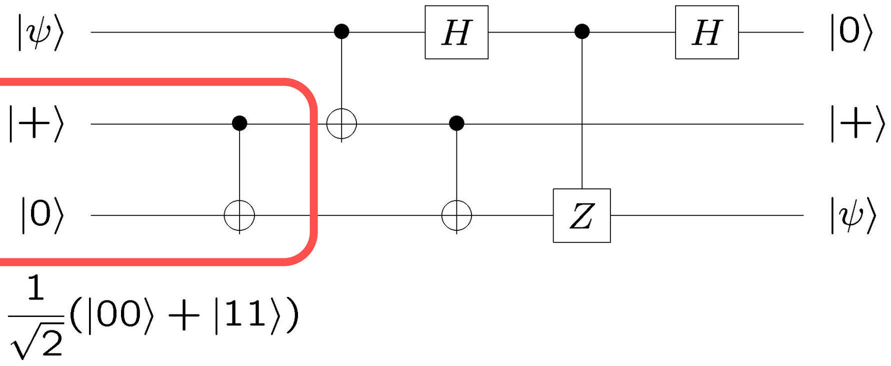
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Deriving Quantum Teleportation

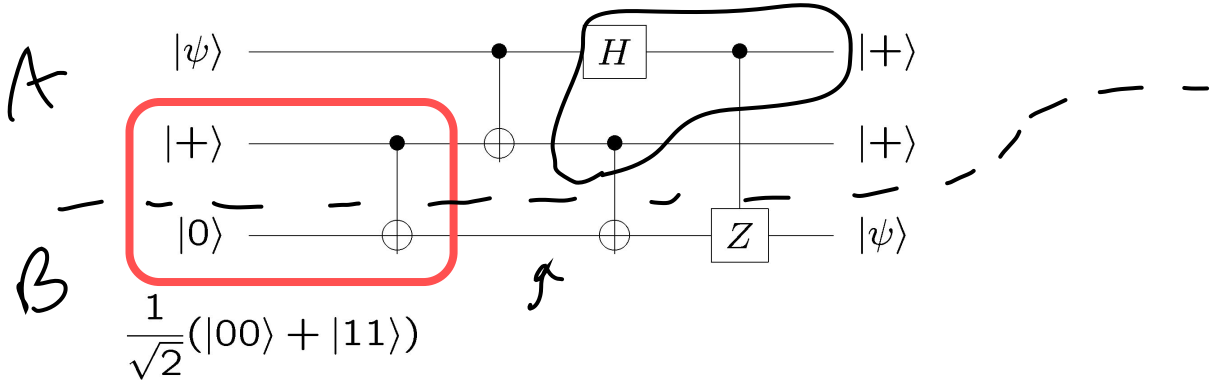


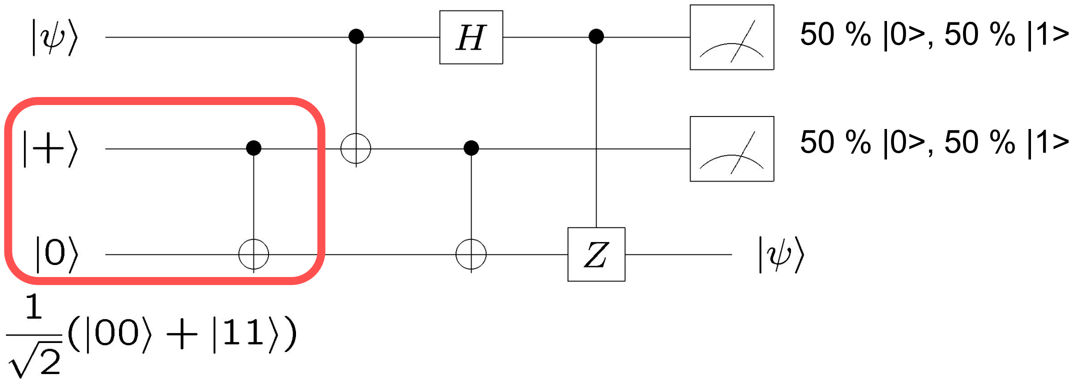
$H|\psi\rangle = |0\rangle$

$H\ |+\rangle = |0\rangle$



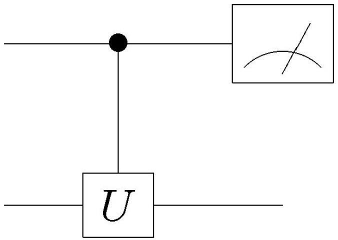
Deriving Quantum Teleportation

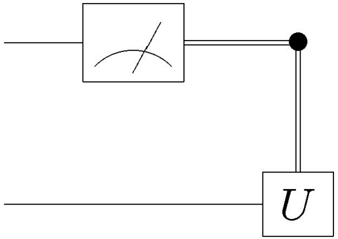




Measurements Through Control

Measurement in the computational basis commutes with a control on a controlled unitary.





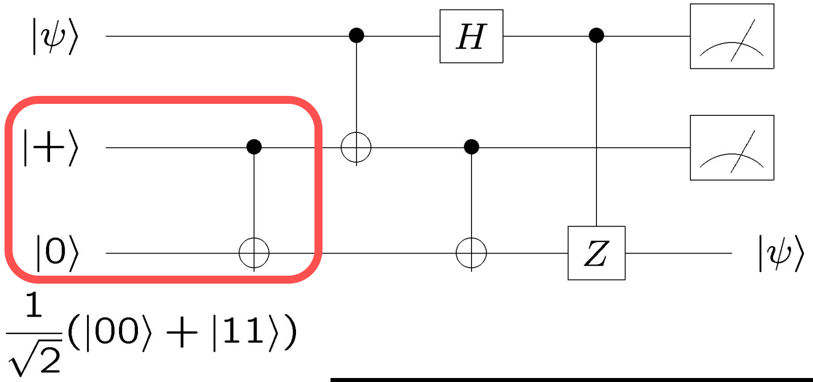
$$(P_0 \otimes I)C_U = C_U(P_0 \otimes I)$$

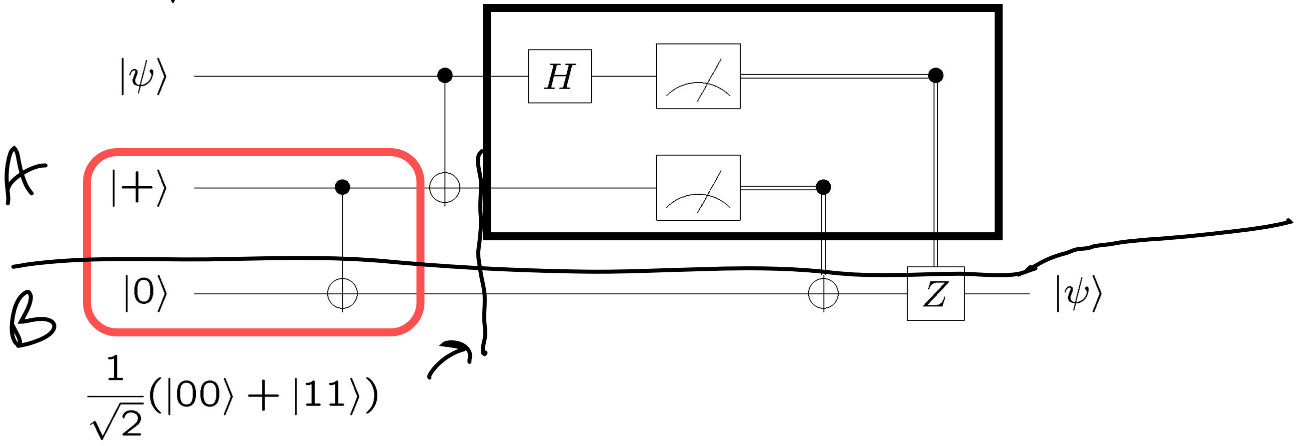
$$(|0\rangle\langle 0|\otimes I)(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U) = (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U)(|0\rangle\langle 0|\otimes I)$$

$$(P_1 \otimes I)C_U = C_U(P_1 \otimes I)$$

$$(|1\rangle\langle 1|\otimes I)(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U) = (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes U)(|1\rangle\langle 1|\otimes I)$$

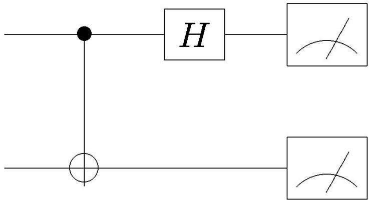
Deriving Quantum Teleportation



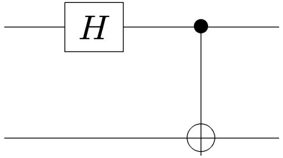


Bell Basis Measurement

Unitary followed by measurement in the computational basis is a measurement in a different basis.



Run circuit backward to find basis:



$$|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

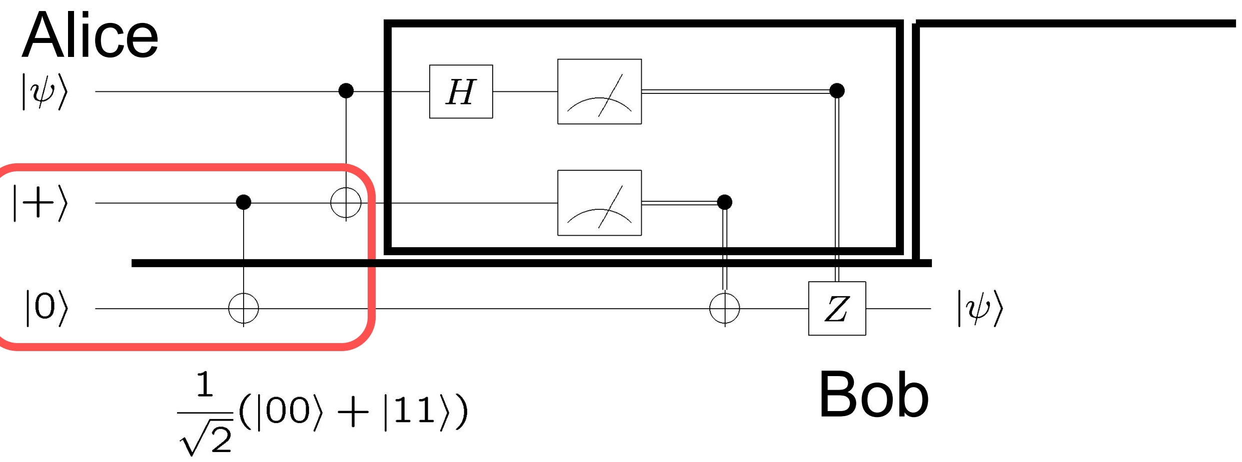
$$|01\rangle \rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|10\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|11\rangle \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Teleportation – this is the quantum circuit

1. Initially Alice has $|\psi\rangle$ and they each have one of the two qubits of the entangled wave function $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
2. Alice measures $|\psi\rangle$ and her half of the entangled state in the Bell Basis.
3. Alice send the two bits of her outcome to Bob who then performs the appropriate X and Z operations to his qubit.

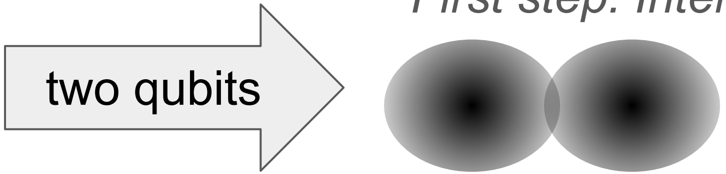


Teleportation - Digression 1

Alice and Bob each have a qubit

- and the wave function of their two qubits is entangled
- we can't think of Alice's qubit as having a particular wave function
- we have to talk about the "global" two qubit wave function.

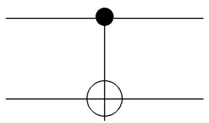
First step: Interact and entangle



Second step: Separate (physically)

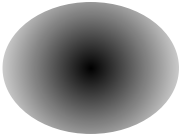
$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|0\rangle$$

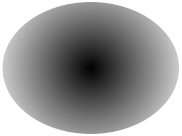


$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Teleportation - Digression 2



$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



Alice does not know the wave function

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

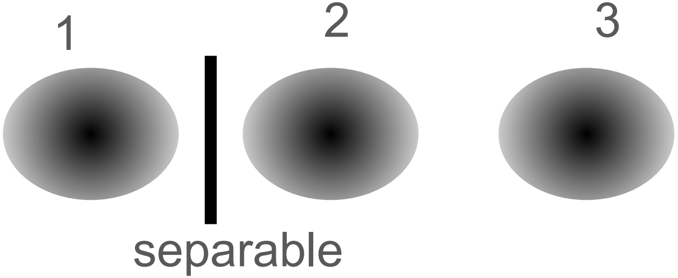
We have three qubits whose wave function is

qubit 1 → $$|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$ ← qubit 2 and qubit 3

Separable, Entangled, 3 Qubits - Digression 3

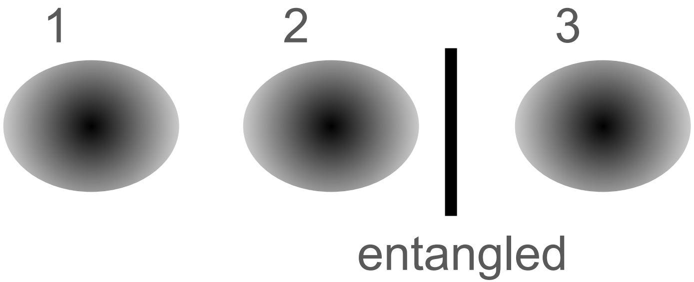
If we consider $(\alpha|0\rangle+\beta|1\rangle)\otimes\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = \frac{\alpha}{\sqrt{2}}|000\rangle+\frac{\alpha}{\sqrt{2}}|011\rangle+\frac{\beta}{\sqrt{2}}|100\rangle+\frac{\beta}{\sqrt{2}}|111\rangle$

- qubit 1 as one subsystem
- qubits 2 and 3 as another subsystem
- then the state is separable across this divide

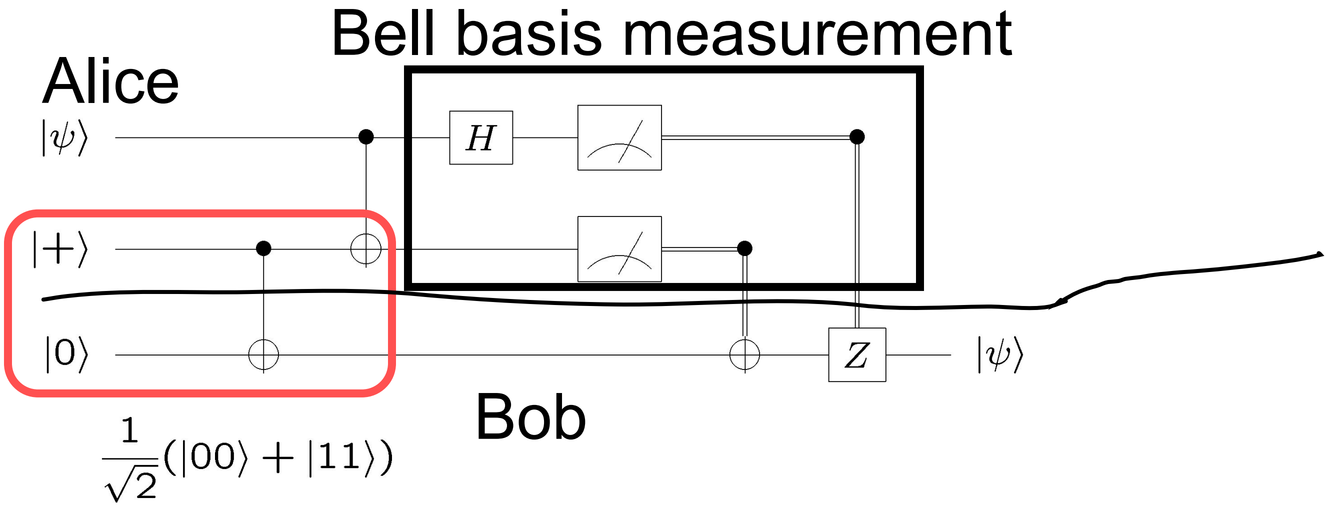


If we consider $\neq (a_{00}|00\rangle+a_{01}|01\rangle+a_{10}|10\rangle+a_{11}|11\rangle)\otimes(b_0|0\rangle+b_1|1\rangle)$

- qubits 1 and 2 as one system
- qubits 3 as one subsystem
- then the state is entangled across this divide.



Teleportation



$$|\psi\rangle_1 \otimes \frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23})$$
$$= \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

Teleportation (we will discuss the Bell states later, too)

$$|\psi\rangle_1 \otimes \frac{1}{\sqrt{2}}(|00\rangle_{23} + |11\rangle_{23}) = \frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

Express this state in terms of Bell basis for first two qubits.

Bell basis

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Computational basis

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle)$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)$$

Teleportation

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle)$$
$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle - |\Psi_-\rangle)$$

$$\frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

$$= \frac{1}{2}(\alpha(|\Phi_+\rangle + |\Phi_-\rangle) \otimes |0\rangle + \alpha(|\Psi_+\rangle + |\Psi_-\rangle) \otimes |1\rangle$$
$$+\beta(|\Psi_+\rangle - |\Psi_-\rangle) \otimes |0\rangle + \beta(|\Phi_+\rangle - |\Phi_-\rangle) \otimes |1\rangle$$

$$= \frac{1}{2}\Big[|\Phi_+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |\Phi_-\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)$$
$$+|\Psi_+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |\Psi_-\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\Big]$$

Teleportation

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$|\Psi_+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

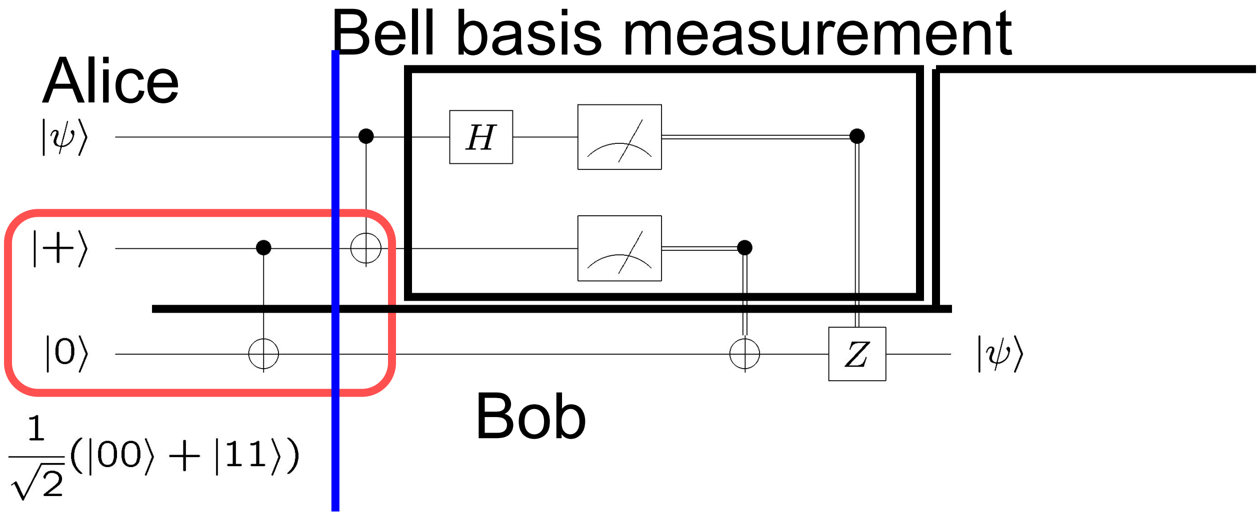
$$|00\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle + |\Psi_-\rangle)$$
$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle)$$
$$|10\rangle = \frac{1}{\sqrt{2}}(|\Psi_+\rangle - |\Psi_-\rangle)$$

$$\frac{\alpha}{\sqrt{2}}|000\rangle + \frac{\alpha}{\sqrt{2}}|011\rangle + \frac{\beta}{\sqrt{2}}|100\rangle + \frac{\beta}{\sqrt{2}}|111\rangle$$

$$= \frac{1}{2}(\alpha(|\Phi_+\rangle + |\Phi_-\rangle) \otimes |0\rangle + \alpha(|\Psi_+\rangle + |\Psi_-\rangle) \otimes |1\rangle$$
$$+\beta(|\Psi_+\rangle - |\Psi_-\rangle) \otimes |0\rangle + \beta(|\Phi_+\rangle - |\Phi_-\rangle) \otimes |1\rangle$$

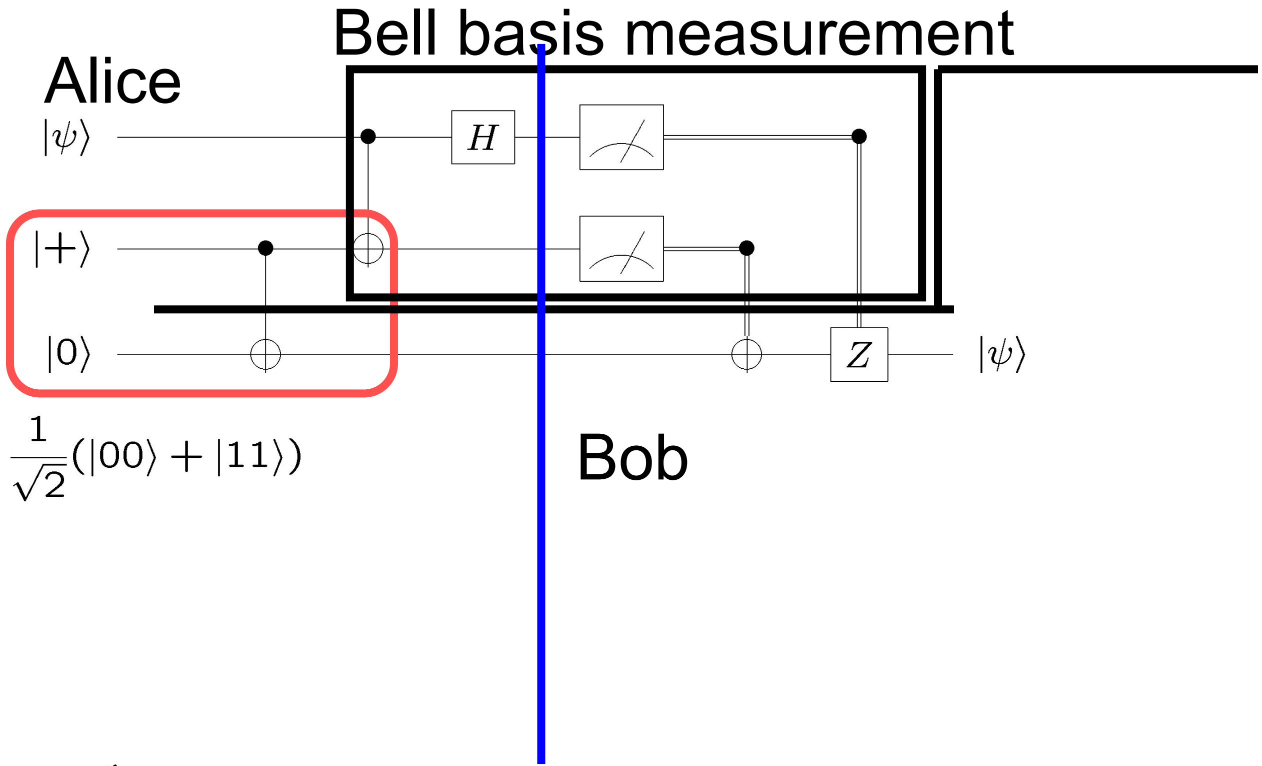
$$= \frac{1}{2}\Big[|\Phi_+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) + |\Phi_-\rangle \otimes (\alpha|0\rangle - \beta|1\rangle)$$
$$+|\Psi_+\rangle \otimes (\alpha|1\rangle + \beta|0\rangle) + |\Psi_-\rangle \otimes (\alpha|1\rangle - \beta|0\rangle)\Big]$$

Teleportation



$$\frac{1}{2}\Big[|\Phi_+\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |\Phi_-\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$+|\Psi_+\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |\Psi_-\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)\Big]$$

Teleportation



$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\frac{1}{2}\left[|00\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |10\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)\right.$$
$$\left. + |01\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |11\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)\right]$$

Teleportation … And now measure two of the qubits

Given the wave function

$$\frac{1}{2}[|00\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |10\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$+|01\rangle_{12}(\alpha|1\rangle_3 + \beta|0\rangle_3) + |11\rangle_{12}(\alpha|1\rangle_3 - \beta|0\rangle_3)]$$

Measure the first two qubits in the computational basis

$$M_{00} = |00\rangle\langle 00| \otimes I \qquad M_{01} = |01\rangle\langle 01| \otimes I$$
$$M_{10} = |10\rangle\langle 10| \otimes I \qquad M_{11} = |11\rangle\langle 11| \otimes I$$

Equal ¼ probability for all four outcomes and new states are:

$$|00\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) \qquad |10\rangle_{12} \otimes (\alpha|0\rangle_3 - \beta|1\rangle_3)$$
$$|01\rangle_{12} \otimes (\alpha|1\rangle_3 + \beta|0\rangle_3) \qquad |11\rangle_{12} \otimes (\alpha|1\rangle_3 - \beta|0\rangle_3)$$

Teleportation … And now measure two of the qubits

If the bits sent from Alice to Bob are 00, do **nothing**

$$|00\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) = |00\rangle_{12} \otimes |\psi\rangle_3$$

If the bits sent from Alice to Bob are 01, apply a **bit flip**

$$(I_4 \otimes X)|01\rangle_{12} \otimes (\alpha|1\rangle_3 + \beta|0\rangle_3) = |01\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta_1|1\rangle_3)$$

If the bits sent from Alice to Bob are 10, apply a **phase flip**

$$(I_4 \otimes Z)|10\rangle_{12} \otimes (\alpha|0\rangle_3 - \beta|1\rangle_3) = |10\rangle \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) = |10\rangle_{12} \otimes |\psi\rangle_3$$

If the bits sent from Alice to Bob are 11, apply **a bit & phase flip**

$$(I_4 \otimes Z)(I_4 \otimes X)|11\rangle_{12} \otimes (\alpha|1\rangle_3 - \beta|0\rangle_3) = (I_4 \otimes Z)|11\rangle_{12} \otimes (\alpha|0\rangle_3 - \beta|1\rangle_3)$$

$$= |11\rangle_{12} \otimes (\alpha|0\rangle_3 + \beta|1\rangle_3) = |11\rangle_{12} \otimes |\psi\rangle_3$$

Discussion for the break…

We can teleport states. Can we teleport quantum gates?