Practical Quantum Computing

Lecture 04
Superdense Coding and Bell’s Inequalities

with slides from Dave Bacon https://homes.cs.washington.edu/~dabacon/teaching/siena/
based on *Quantum Computing:Lecture Notes* by Ronald de Wolf https://homepages.cwi.nl/~rdewolf/qcnotesv2.pdf

Week	**Tuesday** (3h)			**Wednesday** (3h)			**Deadlines**	
1. The Basics	Introduction	Gates	Circuit Identities	Qiskit	Cirq/Qualtran	Q&A		
	Programming Assignment 1: The basics of a quantum circuit simulator			**Programming Assignment 1:** The building blocks of a quantum circuit simulator				
2. Entanglement and its Applications	Teleportation	Superdense Coding	Quantum Key Distribution	Qualtran/ Assignment2	Terminology of Projects	Q&A		
	Programming Assignment 2: The basics of a quantum circuit optimizer			**Programming Assignment 2:** The building blocks of a quantum circuit optimizer				
3. Computing	Phase Kickback and Toffoli	Distinguishing quantum states and The First Algorithms	Grover's Algorithm	Invited TBA	PennyLane	Q&A		11 May 2024
4. Advanced Topics*	Arithmetic Circuits*	Fault-Tolerance*	QML*	Invited TBA	Crumble	Q&A	18 May 2024	

* not evaluated

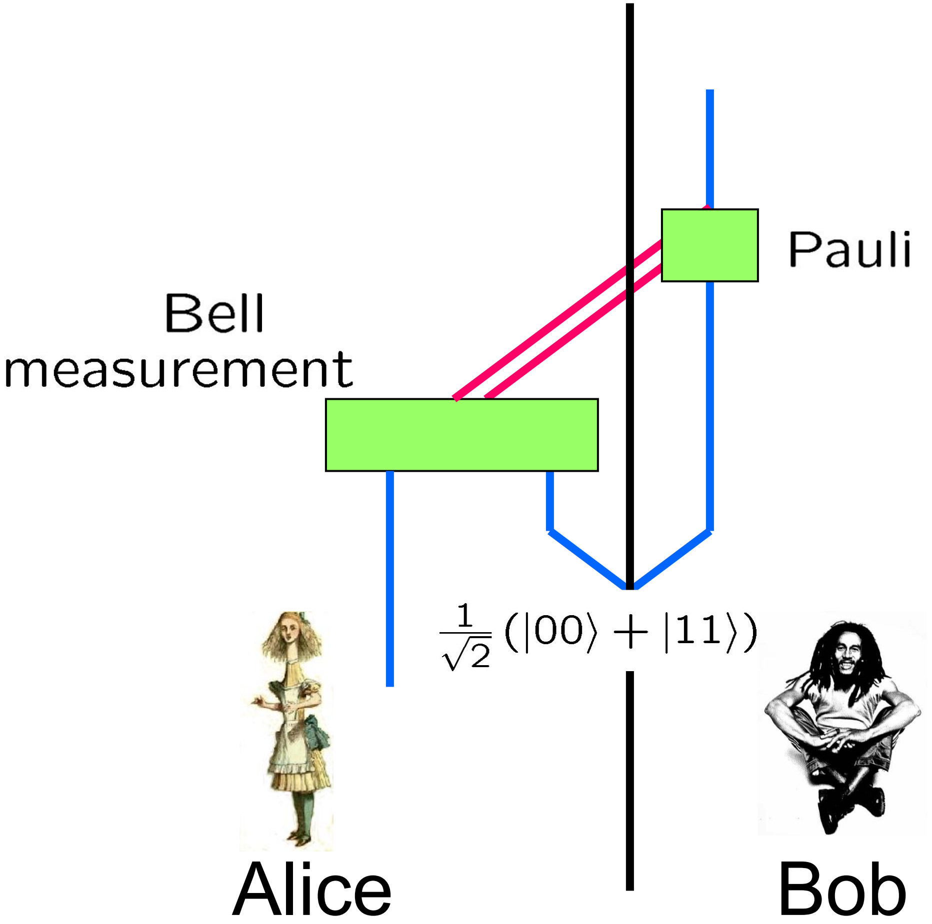
Learning goals - 05 Superdense Coding (Entanglement)

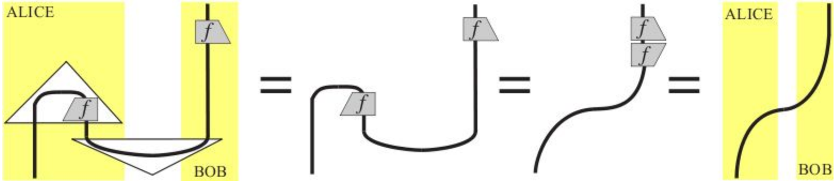
1. What you have learned by now
   a. Quantum circuits: mathematics, diagrams and circuit identities
   b. Teleportation: derivation from circuit identities, using entangled Bell states
2. **Sending two bits of information using entanglement**
   a. What is superdense coding (sdc)?
   b. What is the difference between sdc and teleportation?
3. **Entanglement is more powerful than classical correlations**
   a. Proof of power – using Bell States to show Bell's inequalities
   b. (Appendix*) Winning games using entanglement – quantum games

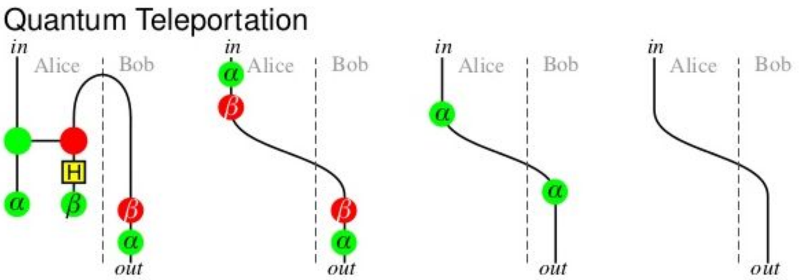
In the exercise session and programming assignment of this week

- basics of quantum circuit optimization
- build our own quantum circuit optimizer
- benchmark your optimizer

Teleportation







https://pennylane.ai/qml/demos/tutorial_zx_calculus/

Teleportation and Superdense Coding

1 qubit = 1 ebit + 2 bits

Teleportation says we can replace transmitting a qubit with a shared entangled pair of qubits plus two bits of classical communication.

Superdense Coding

Next we will see that

2 bits = 1 qubit + 1 ebit

Superdense Coding

Suppose Alice and Bob each have one qubit and the joint two qubit wave function is the entangled state

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

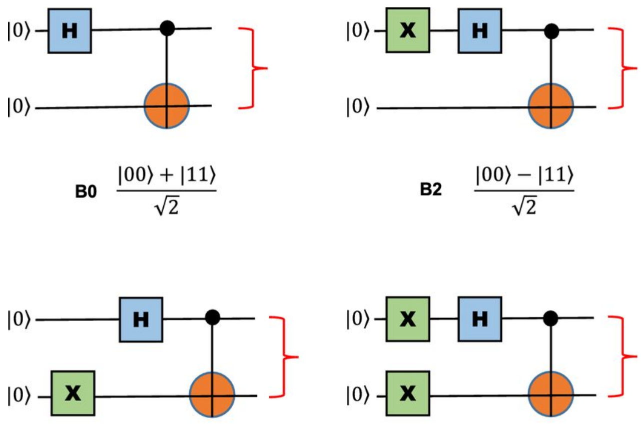
Alice wants to send two bits to Bob. Call these bits b_1 and b_2.

Alice applies the following operator to her qubit:

Bob then measures in the Bell basis to determine the two bits.

2 bits = 1 qubit + 1 ebit

Bell States



B0 $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

B2 $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$

B1 $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$

B3 $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$

B_0 very often called a Bell pair state

B_1 flip the second state |0> to |1>

B_2 flip the phase from + to -

B_3 the spin singlet state from the previous slide

- B_0, B_1, B_2 are also invariant if transformed according to their relation to B_3
- For example, for B_0 considering the observables Z and X:
  - Alice measures in Z and sees |0>
  - the state on Bob's side is |0>
    - measures X (rotated basis)
    - sees with equal probability |+> or |->

Bell Basis

The four Bell states

- can be turned into each other
- using operations on only one of the qubits:

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$(X\otimes I)|\Phi_+\rangle = (X\otimes I)\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = \frac{1}{\sqrt{2}}(|10\rangle+|01\rangle) = |\Psi_+\rangle$$

$$(Z\otimes I)|\Phi_+\rangle = (Z\otimes I)\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle-|11\rangle) = |\Phi_-\rangle$$

$$(ZX\otimes I)|\Phi_+\rangle = (ZX\otimes I)\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = \frac{1}{\sqrt{2}}(-|10\rangle+|01\rangle) = |\Psi_-\rangle$$

Superdense Coding

Initially: $|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Alice applies the following operator to her qubit: $Z^{b_2}X^{b_1}$

$$(Z^{b_2}X^{b_1} \otimes I)|\Phi_+\rangle$$

Bob can uniquely:

- determine which of the four states he has
- figure out Alice's two bits!

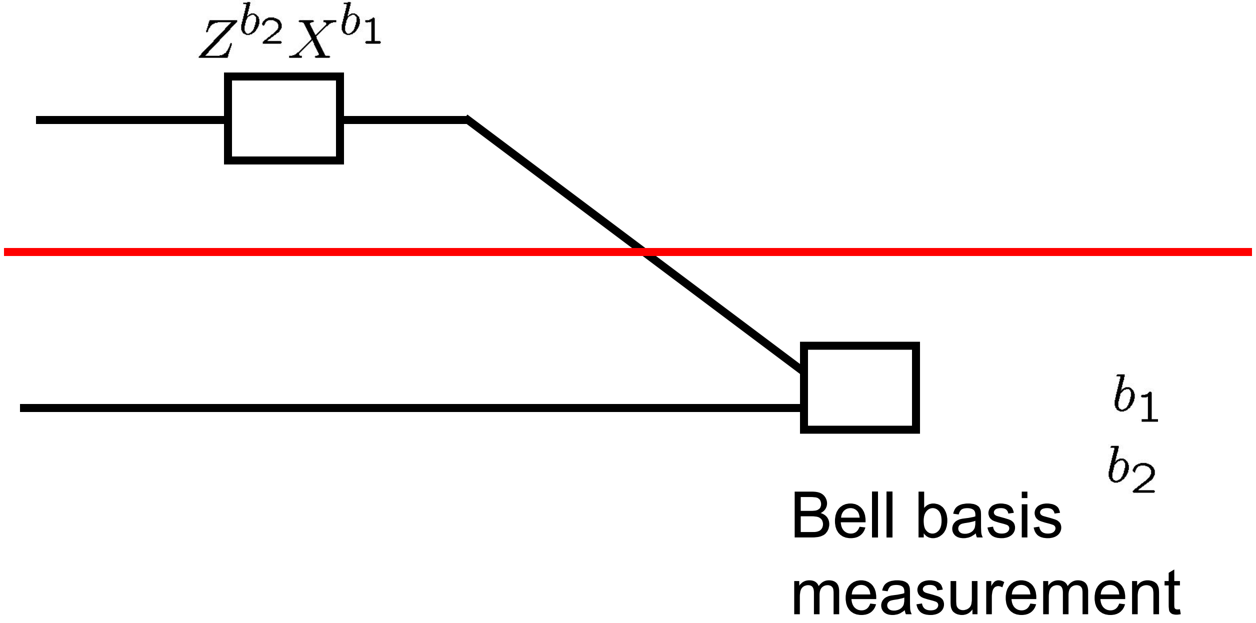
$b_1 = 0, b_2 = 0$	$\lvert\Phi_+\rangle$
$b_1 = 0, b_2 = 1$	$(Z \otimes I)\lvert\Phi_+\rangle = \lvert\Phi_-\rangle$
$b_1 = 1, b_2 = 0$	$(X \otimes I)\lvert\Phi_+\rangle = \lvert\Psi_+\rangle$
$b_1 = 1, b_2 = 1$	$(ZX \otimes I)\lvert\Phi_+\rangle = \lvert\Psi_-\rangle$

Superdense Coding

b_1

b_2

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$



Teleportation and Superdense Coding

1 qubit = 1 ebit + 2 bits

Teleportation says we can replace transmitting a qubit with a shared entangled pair of qubits plus two bits of classical communication.

2 bits = 1 qubit + 1 ebit

Superdense coding. We can send two bits of classical information if we share an entangled state and can communicate one qubit of quantum information.

Learning goals - 05 Superdense Coding (Entanglement)

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2. **Sending two bits of information using entanglement**
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Bell's Inequalities

Is there a set of instructions that tells the particles how to react when they are measured?

Bell Inequalities are a test for *locality* by considering the correlations between measurement outcomes obtained by two parties who share an entangled state

- classical correlations
  - encoded in a set of instructions using hidden variables with known values
  - there is a joint probability distribution that governs the possible outcomes of all measurements
  - then the outcome of any measurement can be predicted with certainty
- quantum correlations

One possible approach:

- take three binary properties A, B and C
- model classic probabilistic behaviour by an inequality
- test on multiple quantum states by collecting statistics

Classical: Count the number of events satisfying a condition

- Assume that the binary properties are randomly measured
- A = {+1, -1}, B = {+1, -1}, C = {+1,-1}
- Formulate an inequality that is classically correct (see below)
  - N(AB’) = number of times A is +1 and B is -1
  - N(BC’) = number of times B is +1 and C is -1
  - N(AC’) = number of times A is +1 and C is -1

N(AB’) + N(BC’) >= N(AC’)

N(AB’) = N(AB’C) + N(AB’C’), because C can be either +1 or -1

N(BC’)= N(ABC’) + N(A’BC’), because A can be either +1 or -1

N(AC’)=N(ABC’) + N(AB’C’), because B can be either +1 or -1

N(AB’C) + <u>N(AB’C’) + N(ABC’)</u> + N(A’BC’) >= <u>N(ABC’) + N(AB’C’)</u>

N(AB’C) + N(A’BC’) >= 0 → it is correct, sum of two positive values > 0

Quantum: Validate experimentally by measuring repeatedly

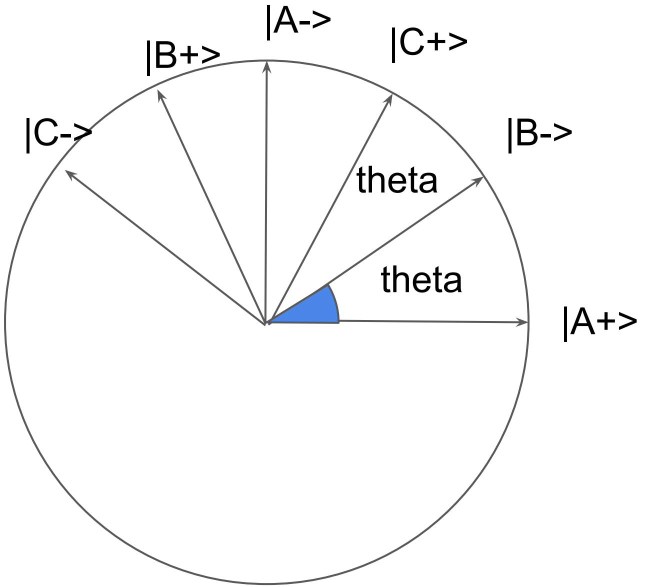
Given an ensemble of entangled states, for example, B_0

- Three axis: Z and two others rotated by angle *theta* and *2theta*
- Alice and Bob ***randomly choose along which axis A, B, or C to measure***

cos(angle/2)|A+> ; sin(angle/2)|A-> > $\begin{pmatrix} \cos(\text{angle}/2) \\ \sin(\text{angle}/2) \end{pmatrix}$

N(AB’) + N(BC’) >= N(AC’)

P(A+, B-) + P(B+, C-) >= P(A+, C-)



$\sin^2(\text{theta}/2) + \sin^2(\text{theta}/2) >= \sin^2(\text{theta})$

For small angles **sin(x) = x**

$2 * (\text{theta}^2/4) >= \text{theta}^2$

Violates Inequality

Discussion for the break…

Entanglement has some non-obvious properties, and it can be used for communication purposes. Entanglement is a communication resource. Is entanglement a computational resource, too?

Appendix

Instructions: Hidden Variables -> Counting is the value of N

Before:
1. each experiment
2. the Bell pair is constructed
3. sent to Alice and Bob

The particles decide how to react *locally* to the measurements

→ **instructions**

A	B	C
+	+	+
+	+	-
+	-	+
+	-	-
-	+	+
-	+	-
-	-	+
-	-	-

On next slide: Instead of three properties(e.g. A,B,C) use three devices and a single property

Entangled State

1/sqrt(2) * (|01> - |10>)   often called the *spin singlet state*

whenever the measurement is performed along the Z axis

It is always possible for Alice to predict what Bob's result was

Alice measures |0>, Bob measures |1>

Alice measures |1>, Bob measures |0>

They share a state that **remains invariant if each apply the same unitary transformation**

$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ |0> = a|a> + b|b> and |1> = c|a> + d|b>

Replace |0> and |1>: **1/sqrt(2) (|<u>01</u>> - |<u>10</u>>) = (ad - bc)/sqrt(2)(|<u>ab</u>> - |<u>ba</u>>)**

U is unitary, (ad-bc) is a global phase factor of the form $e^{i(theta)}$

Measurement results in a rotated basis on both qubits will be correlated too

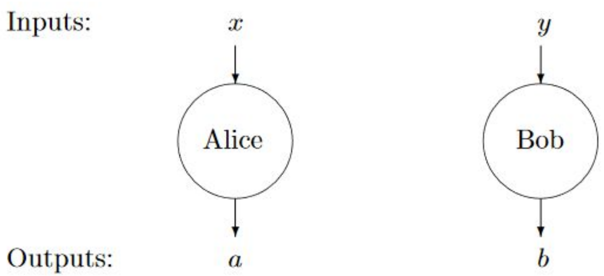
Entanglement and Games

Entangled states

- cannot be written as a tensor product of separate states
- the most famous one is the Bell pair

Non-local games (remember teleportation)

- explore some of the consequences of entanglement
- involves a referee and two *non-communicating* parties
- Alice and Bob are *cooperatively* trying to win the game



The game

- one round of interaction between referee and Alice and Bob:
- the referee sends
  - a (classical) question *x* to Alice
  - a (classical) question *y* to Bob
  - questions are sampled from some known probability distribution
- Alice and Bob respectively respond with a (classical) answer

CHSH Game

CHSH game where two players Alice and Bob

- receive an input bit x and y respectively
- produce an output a and b based on the input bit
- Alice's output bit depends solely on her input bit x, and similarly for Bob

The goal is to maximize the probability to satisfy the condition:

a XOR b = x AND y

Consider the case of classical deterministic strategies

- without any randomness
- *the highest probability achievable is 75%*
- four bits completely characterize any deterministic strategy
    - Let a0, a1 be the outputs that Alice outputs outputs if x=0 and x=1
    - Let b0, b1 be the outputs Bob gives on inputs y=0 and y=1

Not possible to satisfy all four equations simultaneously, since summing them modulo 2 yields **0 = 1**

$$\begin{aligned} a_0 \oplus b_0 &= 0, \\ a_0 \oplus b_1 &= 0, \\ a_1 \oplus b_0 &= 0, \\ a_1 \oplus b_1 &= 1. \end{aligned}$$

CHSH Game

With quantum correlations

- it can achieve higher success probability
- two players start with a shared Bell-pair entangled state
- the **random input x and y** is provided by referee for Alice and Bob

The success probability of satisfying the above condition will be **cos(theta/2)^2** if Alice and Bob measure their entangled qubit in measurement basis V and W where angle between V and W is theta.

Maximum success probability is

- **cos(pi/8)^2 ~ 85.3%** when theta = pi/4.
- In the usual implementation, Alice and Bob share the Bell state with the same value and opposite phase. If the input x (y) is 0, Alice (Bob) rotates in Y-basis by angle -pi/16 and if the input is 1, Alice (Bob) rotates by angle 3pi/16

CHSH Game

What Alice does:

- if $x=0$ then Alice applies $R(-\pi/16)$ to her qubit
- if $x=1$ she applies $R(3\pi/16)$
- then Alice measures her qubit in the computational basis
- outputs the resulting bit a

Bob's procedure is the same, depending on his input bit y

After the measurements

- the probability that $a \oplus b = 0$ is $\cos(\theta_1+\theta_2)^2$
- the first condition is satisfied with probability $\cos(\pi/8)^2$ for all four input possibilities

CHSH Game

Start with

$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

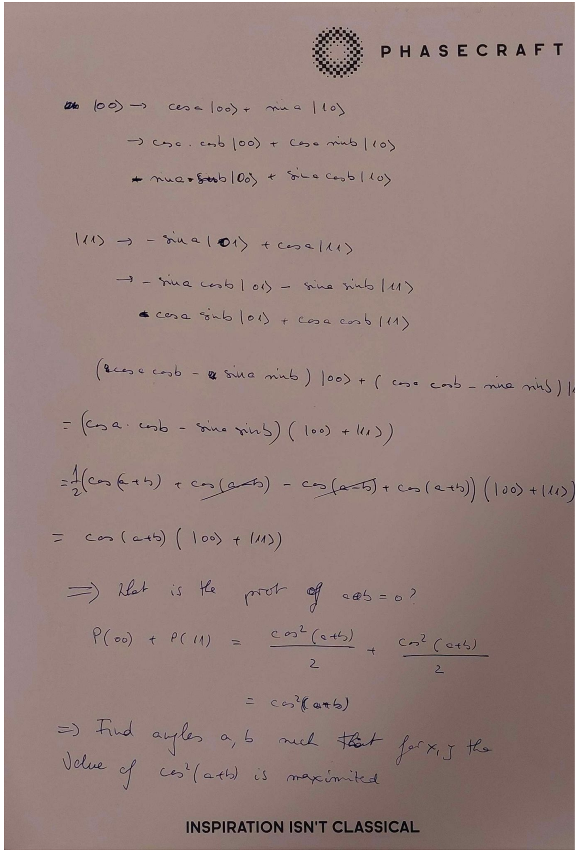
Consider the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

After Alice uses theta1 and Bob uses theta2

$$\frac{1}{\sqrt{2}}\left(\cos(\theta_1+\theta_2)(|00\rangle - |11\rangle) + \sin(\theta_1+\theta_2)(|01\rangle + |10\rangle)\right).$$

Product-to-sum[32]
$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$
$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$
$2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$
$2\cos\theta\sin\varphi = \sin(\theta+\varphi) - \sin(\theta-\varphi)$
$\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{\cos(\theta-\varphi) + \cos(\theta+\varphi)}$
$\prod_{k=1}^{n}\cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1 + \cdots + e_n\theta_n)$ where $S = \{1, -1\}^n$



https://github.com/qiskit-community/qiskit-community-tutorials/blob/master/awards/teach_me_qiskit_2018/chsh_game/CHSH%20game-tutorial.ipynb