Information Economics Final Exam

May 8th, 2023

Daniel Hauser

The exam is 2 hours and has a total of 120 points. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Warm up

   (a) We saw that when a risk neutral manager is contracting with a risk averse agent, in the absence of moral hazard the manager takes on all risk and pays a deterministic wage. Explain, informally, how and why this result changes in the presence of moral hazard in the model we studied in class.

   Solution. *Without moral hazard, in the contracting setting we studied in class, the manager can incentivize effort with a deterministic wage, as they can directly condition the wage on the action chosen. Since the worker is risk averse, the utility an agent would receive from a random wage could be delivered at less cost to the principal with a deterministic wage, implying that the principal takes on all the risk in the optimal contract.*

   Moral hazard means that the action the worker chooses isn't directly observable. A deterministic wage then cannot deliver incentives for effort (with the exception of the lowest cost level of effort), since under a deterministic wage the worker would always choose the cheapest level of effort. In order to make more expensive levels of effort incentive compatible, the firm must reward different levels of output differently, for instance by paying higher wages after output that is indicative of high effort. The optimal wage now depends on the likelihood ratio, which captures how likely different levels of output are under different levels of effort.

(b) Can you sketch an example of a signaling game where an equilibrium that fails the intuitive criterion pareto dominates an equilibrium that satisfies it (an informal description of the game and clear enough graphical justification is enough)?

Solution. *The intuitive criterion is an equilibrium refinement in signaling games. We say an equilibrium fails to satisfy the intuitive criterion if some a types could, after sending some message m argue that (i) another set of would never send that message, as there are no possible beliefs that would improve their payoff after sending that message relative to their equilibrium payoff, therefore (ii) my type must not belong to that set of types, so the receiver should best respond to some belief that doesn't include types from that set. If this would then improve that sender type's payoff, we say that the equilibrium satisfies the intuitive criterion.*

In the two type signaling game we studied in class with unproductive education, only separating equilibrium satisfied the intuitive criterion. But, for a high enough prior, there's a pooling equilibrium that pareto dominates any separating equilibrium.

2. A sender has type $\theta \in \{1, 4\}$. Both types are equally likely. The firm knows their type and they can choose to disclose their type or not ($m \in \{\theta, \emptyset\}$) to a receiver who then chooses action x to maximize utility $u(x, \theta) = -(x - \theta)^2$. The sender receives utility x from the receivers action.

(a) Argue that all information is revealed in every PBE of this game.

Solution. *Throughout this question, I'll let m denote the sender's strategy, μ be the belief that $\theta = 4$, e.g. $\mu(m) = Pr(\theta = 4|m)$, and let $m \in \{1, 4, \emptyset\}$ denote the three possible messages. If either type discloses their type in equilibrium, then all information is revealed. Suppose neither type discloses with probability* 1*. Then, after seeing no disclosure, the receiver forms belief $Pr(\theta = 4|m = \emptyset) < 1$.*

Moreover, following no disclosure, x solves

$$x = \max E(-(x-\theta)^2 | m = \emptyset)$$

So $x = E(\theta | m = \emptyset) < 4$. But, if the high type discloses their type, in equilibrium the receiver must believe that $\theta = 4$, so they receive a payoff of 4 from disclosing. Therefore, at least one type must reveal their type in equilibrium, in which case, both types effectively disclose their type. In any equilibrium of this game, the high type discloses their type with probability 1, and the low type effectively discloses their type, as the receiver beliefs that their type is $\theta = 1$ following both no disclosure and $m = 1$.

Now assume the low type firm can secretly pay a cost of 1 to fabricate evidence that they are a high type. If they pay this cost, then they can also disclose that their type is $\theta = 4$.

(b) Is there a separating equilibrium in this game?

Solution. *Suppose this game had a separating equilibrium. Let $m(4)$ be a message that the high type sends and $m(1)$ be a message that the low type sends in equilibrium. The low type receives a payoff of 1 from sending $m(1)$. If they deviate and send $m(4)$, they receive a payoff of at least $4 - 1_{\{m(4)=4\}} \geq 3$, as the receiver believes that only the high type sends $m(4)$ and the low type can send any message the high type sends. So separation is impossible.*

(c) Is there an equilibrium where both types report $\theta = 4$ with probability 1?

Solution. *Let equilibrium beliefs be as follows. After both non-disclosure and $m = 1$, the receiver believes that the sender is type $\theta = 1$. Then, after seeing $\theta = 4$, by Bayes rule the receiver plays $E(\theta) = \frac{1}{2}4 + \frac{1}{2} = \frac{5}{2}$. So, high type receives $5/2$ from sending $m = 4$, and receives 1 from any other message. The low type receives $3/2$ from $m = 4$ and 1 from any other message. So this is an equilibrium.*

(d) Is there an equilibrium where both types send message $\emptyset$ with probability 1?

Solution. *If both types pool on the empty set, they receive a payoff of* $5/2$. *Since only the low type can send* $m = 1$, *the receiver pays* 1 *after seeing* $m = 1$, *so the low type doesn't want to deviate and disclose. Since both types can play* $m = 4$, *PBE (and sequential equilibrium), don't place any restrictions on beliefs that can form after seeing* $m = 4$, *so as long as the receiver beliefs that* $Pr(\theta_2|m = 4) \leq 1/2$, *then neither type has incentive to deviate. So there is a pooling equilibrium with the desired structure.*

(e) Can you construct a mixed strategy equilibrium where $\theta = 1$ randomizes between fabricating a signal and not fabricating one.

Solution. *Let's look for an equilibrium where* $\theta = 4$ *reveals their type for sure, and assume that the the receiver interprets any other message as proof that the sender is the low type. So* $E(\theta|m = 1) = E(\theta|m = \emptyset) = 1$. *In order to get the incentives to mix, it must be that* $E(\theta|m = 4) - 1 = 1$. *So*

$$4Pr(\theta = 4|m = 4) + (1 - Pr(\theta = 4|m = 4)) = 2$$

So $Pr(\theta = 4|m = 4) = 1/3$. *We can then use Bayes rule to back out the mixed strategy as*

$$Pr(\theta = 1|m = 4) = \frac{Pr(m = 4|\theta = 1)Pr(\theta = 1)}{Pr(m = 4)} = \frac{Pr(m = 4|\theta = 1)}{Pr(m = 4|\theta = 1) + 1}$$

so $Pr(m = 4|\theta = 1) = 1/2$.

3. A government wants a firm to decarbonize its production process. The firm has type θ drawn uniformly from $[0, 1]$ which describes the probability that the firm is able to develop a cleaner process. Specifically, the firm is able to develop a cleaner process by time x with probability $1 - e^{-\theta x}$. For simplicity, assume that this development process is costless for the firm, but the firm also doesn't directly benefit from decarbonizing.

The government wants to take into account the difficulty of decarbonization. So, they design a menu of possible *deadlines* $x \in \mathbb{R}_+$ and a *penalties* $p \in \mathbb{R}_+$, where if the firm has not adopted the cleaner process at time x they must pay a fine p, after paying this they are no longer required to decarbonize. So the firm's expected payoff is $e^{-\theta x} p$ if they face deadline x and penalty p.

At time 0, the firm can relocate in another country where they do not have to decarbonize, but this costs them $-K$ for some $K > 0$. The firm is risk neutral, patient (so they don't discount), and is unable to relocate after time 0.

(a) The government wants to design a deadline and penalty scheme that satisfy the following constraints:

$$
\begin{aligned}
&- p(\theta)e^{-\theta x(\theta)} \geq -p(\theta')e^{-\theta x(\theta')} \ \forall \theta, \theta' \in [0,1] \\
&- p(\theta)e^{-\theta x(\theta)} \geq -K \ \forall \theta \in [0,1].
\end{aligned}
$$

Explain in words what each of these constraints captures.

Solution. *The first constraint is the incentive constraint. It says that a firm prefers to report their type truthfully and face the deadline and penalty the government has prescribed for their type as opposed to reporting that they are another type.*

The second constraint is an individual rationality constraint. It captures that the firm should always want to participate in the government's mechanism instead of relocating to another country.

(b) Note that we can rewrite the first constraint as

$$\theta x(\theta) - \ln p(\theta) \geq \theta x(\theta') - \ln p(\theta')$$

Prove that any deadline that satisfies the constraints must be non-decreasing in θ.

Solution. *Consider any pair of types $\theta > \theta'$. We know that, under* *these constraints*

$\theta x(\theta) - \ln p(\theta) \geq \theta x(\theta') - \ln p(\theta') \theta' x(\theta') - \ln p(\theta') \quad \geq \theta' x(\theta) - \ln p(\theta).$

Adding these together gives

$$(\theta - \theta')(x(\theta) - x(\theta')) \geq 0.$$

(c) Show that the constraints imply that

$$\ln p(\theta) = \underline{V} + \theta x(\theta) - \int_0^\theta x(s)\, ds$$

form some $\underline{V} \leq \ln K$. Is the optimal penalty increasing on decreasing?

Solution. *Let* $V(\theta) = \theta x(\theta) - \ln p(\theta)$*. The incentive constraint can then be rewritten as*

$$V(\theta) \geq V(\theta') - \theta' x(\theta') + \theta x(\theta')$$

Applying this to the two θ, θ' *constraints, we get*

$$(\theta - \theta')x(\theta) \geq V(\theta) - V(\theta') \geq (\theta - \theta')x(\theta').$$

Dividing through by $\theta - \theta'$ *and taking limits gives us*

$$V'(\theta) = x(\theta)\, (a.e.).$$

Integrating this then gives

$$V(\theta) = \int_0^\theta x(s)\, ds + V(0)$$

and by individual rationality $V(0) \geq -\ln K$*. Rearranging terms we get*

$$\ln p(\theta) = \theta x(\theta) - \int_0^\theta x(s)\, ds - V(0)$$

which gives us the desired relationship.

(d) Show that any non-decreasing $x(\theta)$ and the corresponding $p(\theta)$ that satisfies the equation in (c) for some $\underline{V} \leq \ln K$ satisfy all the constraints in (a).

Solution. *If we plug the expression for* $\ln p(\theta)$ *into the IC constraint, we get*

$$\int_0^\theta x(s)\,ds \geq (\theta - \theta')x(\theta') + \int_0^{\theta'} x(s)\,ds$$

which, if we rearrange terms gives us

$$\int_{\theta'}^\theta x(s)\,ds \geq (\theta - \theta')x(\theta')$$

We can rewrite this constraint as

$$\int_{\theta'}^\theta (x(s) - x(\theta'))\,ds \geq 0.$$

Since $x(s)$ *is increasing, this holds. So the all IC constraints are satisfied. For the IR constraint, we can see that*

$$-p(0)e^{-0x(0)} = -exp(\underline{V}) \geq -exp(\ln K) = -K$$

So the IR constraint for the lowest type holds. For the higher types, the IC constraint implies this as

$$-p(\theta)e^{-\theta x(\theta)} \geq -p(\theta')e^{-\theta x(\theta')} \geq -p(\theta')e^{-\theta' x(\theta')}$$