Information Economics Final Exam

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The exam is 2 hours and has a total of 120 points. Please answer as many questions as you can. Answer shortly but justify your answers and explain accurately what you are doing. If you are confused about some question statement, please explain clearly what you assume when answering. Point totals reflect the difficulty of the problem and give a rough estimate for how long the question should take.

1. Warm up

   (a) *(10 points)* In our canonical principal-agent model, describe the optimal contract when a risk neutral agent is contracting with a risk neutral principal. How does this contract change under limited liability.

   Solution. *When two risk neutral parties contract, the principal can induce the optimal level of effort by "selling the firm". They pay the agent a wage equal to their output, and charge them a flat fee equal to the expected output of an agent under the optimal level of effort. With limited liability this is no longer possible, since they can't pay the agent a negative wage. The first best is no longer achievable, and in the example we saw in class this led to a contract that induced lower effort than the first best.*

   (b) *(10 points)* What is the revelation principle. Briefly describe its role in mechanism design.

   Solution. *Informally, the revelation principle says that any equilibrium outcome of any mechanism can also be implemented as the outcome of a direct mechanism – where agents report their types to the principle. This allows us to work with direct mechanisms, which really simplifies the optimization problem – agents actions can be taken to be type reports and to check incentive compatibility we just need to check if truthful reporting is optimal.*

2. Consider the following signaling game. There are three equally likely sender types $\theta \in \{1, 2, 3\}$. The sender chooses message $e \in \mathbb{R}_+$ and a receiver chooses action $x \in \mathbb{R}_+$. The receiver receives utility from choosing action x when facing a type θ receives of $u(x, \theta) = -(x - \theta)^2$, and the sender receives utility $u(x, \theta) = x - \frac{1}{\theta} e^5$. Let $e(\theta)$ be the sender's strategy.

   Describe a perfect Bayesian equilibrium (both strategies and beliefs) consistent with each of the following sender strategies, and verify that it is a PBE.

   (a) *(5 points)* $e(1) = e(2) = e(3) = 0$.

   Solution. *The sender plays the above strategy. The receiver plays* $x = 2$, *and plays* $x = 1$ *otherwise. Beliefs are* $\mu(\theta = 1|e = 0) = \mu(\theta = 2|e = 0) = \mu(\theta = 3|e = 0) = 1/3$ *and* $\mu(\theta = 1|e) = 1$ *for all* $e \neq 0$. *Receiver optimality is obvious (they play* $E(\theta)$*). Each sender receives* 2 *from playing* $e = 0$, *and receives* $1 - 1/\theta e^5$ *from any other level of effort, so this is an equilibrium.*

   (b) *(10 points)* $e(1) = 0$, $e(2) = 1$, $e(3) = 3^{1/5}$.

   Solution. *As before, for any off path action, place prob 1 on* $\theta = 1$. *After each equilibrium action, place prob* 1 *on the type that plays that action and let* $x = E(\theta)$ *under those beliefs. We need to make sure no player wants to deviate. On path,* $\theta = 1$ *receives* 1, $\theta = 2$ *receives* $2 - 1/2 = 3/2$ *and* $\theta = 3$ *receives* 2. *The best off path deviation is to play* $e = 0$, *which gives a payoff of* 1, *so no one wants to make this deviation. So, we simply have to make sure no one wants to mimic another type.*

   - Type $\theta = 1$ *receives* $2 - 1 = 1$ *from playing* $e = 1$ *and* $3 - 3 = 0$ *from* $e = 3^{1/5}$. *So they have no profitable deviation.*
   - Type $\theta = 2$ *receives* $3 - 3/2 = 3/2$ *from playing* $3^{1/5}$ *so they have no profitable deviation.*
   - Type $\theta = 3$ *receives* $2 - 1/3 = 5/3$ *from playing* $\theta = 3$, *so they have no profitable deviation.*

(c) *(10 points)* $e(1) = e(2) = 0$, $e(3) = 3^{1/5}$.

Solution. *At* $e = 0$*, by Bayes rule,* $\mu(\theta = 1|0) = \mu(\theta = 2|0) = 1/2$*, and* $x = 3/2$*. After* $3^{1/5}$*, beliefs must place probability 1 on* $\theta = 3$ *and* $x = 3$*. Again, for all off path actions let* $\mu(\theta = 1) = 1$ *and* $x = 1$*. Players receiver at most* 1 *from an off path action. On path, as above* $\theta = 3$ *receives* 2*, and* $\theta = 1$ *and* $\theta = 2$ *receive* $3/2$*. If* $\theta = 3$ *plays e* $= 0$ *they get* $3/2 < 2$*, so this is not profitable.* $\theta = 2$ *has stronger incentives to deviate that* $\theta = 1$*. If they play* $3^{1/5}$ *they receive* $3/2$*, so no player has incentive to deviate.*

(d) *(25 points)* Which of the above satisfy the intuitive criterion?

Solution. *The best possible beliefs after any deviation are to place probability* 1 *on* $\theta = 3$*. We can use this to pin down the dominated types in each of the three equilibria. In a, a type is dominated if*

$$3 - \frac{1}{\theta}e^5 < 1$$

so type 1 and 2 are dominated beyond $4^{1/5}$ *and only type 1 is dominated between* $2^{1/5}$ *and* $4^{1/5}$*. If* 3 *deviates to play* $4^{1/5}(+\varepsilon)$ *they would receive a payoff of* $3 - 4/3 > 1$ *if no weight was placed on dominated types, so this equilibrium fails.*

The equilibrium in b passes. We can see that $\theta = 2$ *is dominated for any level of effort above* $3^{1/5}$ *and* $\theta = 1$ *is dominated for any level of effort above* $2^{1/5}$*.* $\theta = 3$ *can't benefit from increasing their effort, nor can* $\theta = 2$ *and if* $\theta = 3$ *reduces their effort to something between* $2^{1/5}$ *and* $3^{1/5}$*, they receive a payoff below the payoff they'd receive from mimicking* $\theta = 2$*.*

In c, we can again look for where type 1 *and* 2 *are dominated. This is when*

$$3 - \frac{1}{\theta}e^5 < 3/2$$

so 1 *is dominated when* $e^5 > 3/2$ *and* 2 *is dominated when* $e^5 > 3$*. Type* 3 *then has no reason to increase their effort. If they decrease it to some e s.t.* $e^5 \in (3/2, 3)$ *then the worst case is that they are* *perceived as type* 2*, in which case they receive at most* $2-1/2 = 3/2$*. So type 3 still cannot benefit. If type 2 increases their effort to* $e^5 \in (3/2, 3)$ *then they receive at most* $2 - 3/4 = 5/4$*. So, this satisfies the intuitive criterion as well.*

3. There are two siblings who each inherit 50% of an apartment. Sibling i values the apartment at $\theta_i \in [0, 1]$. Valuations are independent, and are uniformly drawn from $[0, 1]$. The siblings hire a mediator to design a mechanism to redistribute the shares in the most efficient way possible. Sibling i's payoff from receiving a share $x_i \in [0, 1]$, $x_i + x_{-i} \leq 1$, of the apartment and transfer $t_i \in \mathbb{R}$ is $x\theta_i - t$. If sibling does not participate in the mechanism, they keep their share and receive a payoff of $\theta_i/2$.

   (a) *(10 points)* In this setting, a direct mechanism is described by a feasible allocation of shares $x_i(\theta_i, \theta_{-i})$ and transfers $t_i(\theta_i, \theta_{-i})$ for each sibling. Write down each of the following constraints:

   - Bayesian incentive compatibility
   - Dominant strategy incentive compatibility
   - Interim individual rationality
   - Ex-ante individual rationality

   Solution. *easy*

   (b) *(10 points)* What is the ex-post efficient allocation rule. Construct a Pivot (VCG) mechanism in this setting.

   Solution. *In any ex-post efficient mechanism, the higher type sibling should inherit the apartment. So, in a VCG mechanism*

$$t_1(\theta_1, \theta_2) = \begin{cases} -\theta_2 + h(\theta_2) \ \textit{if}\ \theta_2 > \theta_1 \\ h(\theta_2) \ \textit{if}\ \theta_2 < \theta_1 \end{cases}$$

   With t_2 *defined analogously. A natural VCG mechanism here (if we* *interim want IR to be satisfied) is*

$$t_1(\theta_1, \theta_2) = \begin{cases} -\frac{1}{4}\theta_2 \text{ if } \theta_1 < \theta_2 \\ \frac{3}{4}\theta_2 \text{ otherwise} \end{cases}$$

Defined analogously for 2. This is not budget balanced, but interim IR holds exactly for $\theta = 1/2$

A natural mechanism that requires no subsidy (the pivot mechanism) here is

$$t_1(\theta_1, \theta_2) = \begin{cases} \theta_2 \text{ if } \theta_1 > \theta_2 \\ 0 \text{ otherwise} \end{cases}$$

Note that this isn't individually rational.

(c) *(10 points)* Show that in any ex-post efficient BIC mechanism, the expected transfer, $T_i(\theta_i) = E(t_i(\theta_i, \theta_{-i})|\theta_i)$, for an agent of type θ_i must be $\frac{1}{2}\theta_i^2 + V$ for some constant V.

Solution. *Envelope theorem gives*

$$V_i(\theta_i) - V_i(0) = \int_0^{\theta_i} X(s)\, ds$$

where $X(s) = Pr(s > \theta_{-i}|s) = s$. *So* $V_i(\theta_i) - V_i(0) = \frac{1}{2}\theta_i^2$. *We also know that*

$$V_i(\theta_i) = \theta_i^2 - T(\theta_i).$$

Combining these gives the desired formula.

(d) *(15 points)* The mediator decides to run a second price auction, where the winner pays the loser their bid.[1] This is clearly budget balanced. Show that it is a Bayes Nash equilibrium for bidders to bid $\theta_i/3 + 1/6$.

Solution. *We can use the envelope characterization here. This is BIC iff the induced transfer* $T(\theta_i)$ *satisfies the envelope condition.*

[1] So if sibling one bids 1/2 and sibling two bids 1/4, sibling one receives the apartment and pays sibling two 1/4.

So, we need to calculate

$$\theta_i E(\theta_{-i}/3+1/6|\theta_i > \theta_{-i})-(1-\theta_i)E(\theta_i/3+1/6|\theta_i < \theta_{-i}) = \frac{1}{2}\theta_i^2-1/6$$

Note that if I bid 0 *I get a payoff of* $1/6$*, while if I follow this strategy I receive* $\frac{1}{2}\theta_i^2 + 1/6$*. If I bid* 1*, I receive* $\theta_i - 1/2$*, which is lower.*

(e) *(5 points)* Verify that this auction is interim individually rational. How does this result compare with the Myerson-Satterthwaite Theorem?

Solution. *Type* θ_i*'s interim payoff is*

$$\frac{1}{2}\theta_i^2 + \frac{1}{6}.$$

The IR constraint is

$$\frac{1}{2}\theta_i^2 + \frac{1}{6} \geq \frac{1}{2}\theta_i$$

Note that this is non-monotonic, and in fact is tightest at $\theta_i = 1/2$*. It still holds at* $1/2$*, so this mechanism is individually rational, in fact strictly so (so this same mechanism would be an IR, BIC way of resolving this even with some asymmetry in the outside options). By having a more even share of the initial surplus, we can design an ex-post efficient, budget balanced, individually rational mechanism.*