NORMS

1. $\|\underline{x}\| \geq 0$ and $\|\underline{x}\| = 0 \Leftrightarrow \underline{x} = \underline{0}$

2. $\|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

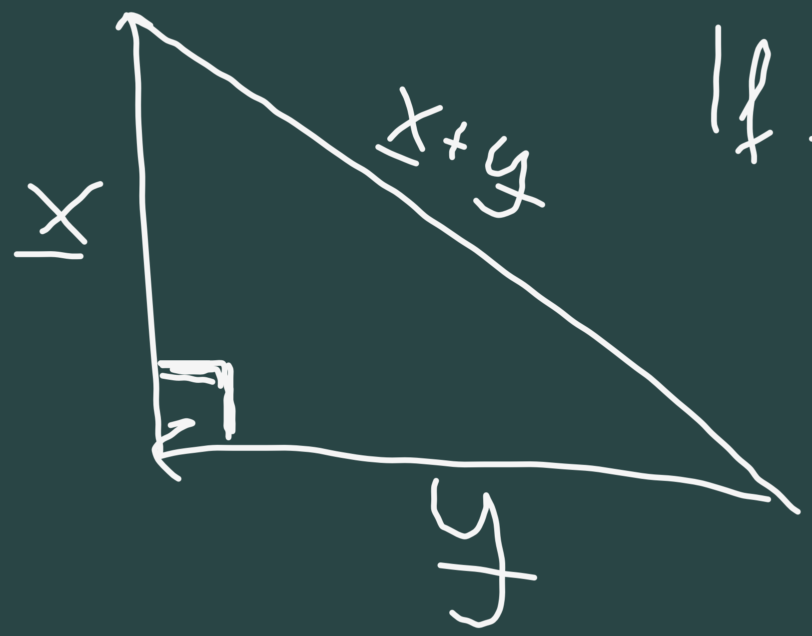
3. $\|\alpha \underline{x}\| = |\alpha| \, \|\underline{x}\|$

INNER PRODUCTS

1. $\langle \underline{x}, \underline{x} \rangle \geq 0$ and $\langle \underline{x}, \underline{x} \rangle = 0 \Leftrightarrow \underline{x} = \underline{0}$

2. $\langle \underline{x}, \underline{y} \rangle = \langle \underline{y}, \underline{x} \rangle$

3. $\langle \alpha \underline{x} + \beta \underline{y}, \underline{z} \rangle = \alpha \langle \underline{x}, \underline{z} \rangle + \beta \langle \underline{y}, \underline{z} \rangle$



If $\underline{x} \perp \underline{y}$ $\quad \underline{x} \cdot \underline{y} = 0$

$$0 = \cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \|\underline{y}\|}$$

DEF

$\underline{x}, \underline{y} \in \mathbb{R}^n$ are ORTHOGONAL wrt $\langle \cdot , \cdot \rangle$ if $\langle \underline{x}, \underline{y} \rangle = 0$

LEMMA

Let $\{q_1, \dots, q_k\} \subset \mathbb{R}^n$ be such that $\langle q_i, q_j \rangle = 0 \ \forall i \neq j$ and $q_i \neq \underline{0} \ \forall i$. Then $\{q_1, \dots, q_k\}$ are linearly independent.

$$\sum_{i=1}^{k} \alpha_i q_i = \underline{0} \qquad \langle q_j, \underline{0} \rangle = \langle q_j, 0 \cdot q_j \rangle = $$

$$\overset{(2)}{=} \langle 0 \cdot q_j, q_j \rangle \overset{(3)}{=} 0 \cdot \langle q_j, q_j \rangle = 0$$

$$0 = \left\langle \sum_{i=1}^{k} \alpha_i q_i, q_j \right\rangle \overset{(3)}{=} \sum_{i=1}^{k} \alpha_i \langle q_i, q_j \rangle = \alpha_j \underbrace{\langle q_j, q_j \rangle}_{\neq 0 \text{ by } (1)}$$

$$\Longrightarrow \alpha_j = 0 \text{ for all } j = 1, \dots, k.$$

$$\Longrightarrow \underline{\alpha} = \underline{0} \Longrightarrow q_1, \dots, q_k \text{ are lin. independent}$$

$$\langle \alpha_1 q_1 + \sum_{i=2}^{k} \alpha_i q_i , q_j \rangle = \alpha_1 \langle q_1, q_j \rangle + \langle \sum_{i=2}^{k} \alpha_i q_i , q_j \rangle$$

$$= \alpha_1 \langle q_1, q_j \rangle + \alpha_2 \langle q_2, q_j \rangle + \langle \sum_{i=3}^{k} \alpha_i q_i , q_j \rangle$$

If $E = \mathrm{span}(q_1, \ldots, q_k)$ as in the Lemma, then

$\{q_1, \ldots, q_k\}$ are a basis

Hence, for all $\underline{x} \in E$, there are unique α_i

s.t. $\underline{x} = \sum_{i=1}^{k} \alpha_i q_i$

$$\langle q_j, \underline{x} \rangle = \sum_{i=1}^{k} \alpha_i \langle q_j, q_i \rangle = \alpha_j \underbrace{\langle q_j, q_j \rangle}_{\neq 0}$$

$$\Longrightarrow \alpha_j = \frac{\langle q_j, \underline{x} \rangle}{\langle q_j, q_j \rangle}$$

IF the basis is ORTHONORMAL

= ORTHOGONAL + $\langle q_j, q_j \rangle = 1$

then $\underline{\alpha}_j = \langle q_j, \underline{x} \rangle$

EXAMPLE 1

$$q_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \quad q_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \quad \dots$$

then $\{q_1, \dots, q_n\}$ are orthonormal wrt DOT PRODUCT

EXAMPLE 2

$$q_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \quad q_2 = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$$

are " " " "

$$\|\underline{x}\|_2 = \left(\sum x_i^2\right)^{\frac{1}{2}} = \left(\underline{x} \cdot \underline{x}\right)^{\frac{1}{2}}$$

LEMMA

Let $\langle \cdot, \cdot \rangle$ be an inner product. Then the function

$$\|\underline{x}\| = \left(\langle \underline{x}, \underline{x} \rangle\right)^{\frac{1}{2}}$$ is a norm.

Moreover, it satisfies the Cauchy-Schwarz inequality:

$$|\langle \underline{x}, \underline{y} \rangle| \leq \|\underline{x}\| \, \|\underline{y}\| \quad \text{for all } \underline{x}, \underline{y} \in \mathbb{R}^n$$

Proof

① $\|\underline{x}\|^2 = \langle \underline{x}, \underline{x} \rangle \geq 0$ and $= 0 \Leftrightarrow \underline{x} = \underline{0}$

③ $\|\alpha \underline{x}\|^2 = \langle \alpha \underline{x}, \alpha \underline{x} \rangle \overset{③}{=} \alpha \langle \underline{x}, \alpha \underline{x} \rangle \overset{②+③}{=} \alpha^2 \langle \underline{x}, \underline{x} \rangle = \alpha^2 \|\underline{x}\|^2$

TRICK

If $\langle \underline{a}, \underline{b} \rangle = 0$ then $\|\underline{a} + \underline{b}\|^2 = \|\underline{a}\|^2 + \|\underline{b}\|^2$

$$\underline{x} = \underbrace{\frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{y}\|^2} \underline{y}}_{\underline{a}} + \underbrace{\left(\underline{x} - \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{y}\|^2} \underline{y} \right)}_{\underline{b}}$$

$$\langle \underline{b}, \underline{a} \rangle = \langle \underline{x}, \underline{a} \rangle - \frac{\langle \underline{x}, \underline{y} \rangle}{\|\underline{y}\|^2} \langle \underline{y}, \underline{a} \rangle =$$

$$= \frac{\langle \underline{x}, \underline{y} \rangle^2}{\|\underline{y}\|^2} - \frac{\langle \underline{x}, \underline{y} \rangle^2}{\|\underline{y}\|^2} = 0$$

$$\|\underline{x}\|^2 = \|\underline{a}+\underline{b}\|^2 \underset{\text{(TRICK)}}{=} \|\underline{a}\|^2 + \|\underline{b}\|^2$$

$$= \frac{\langle \underline{x},\underline{y}\rangle^2}{\|\underline{y}\|^{\cancel{4}\,2}} \cancel{\|\underline{y}\|^2} + \|\underline{b}\|^2 \geq \frac{\langle \underline{x},\underline{y}\rangle^2}{\|\underline{y}\|^2}$$

$\Longrightarrow$ If $\underline{y} \neq 0$, then this implies $\|\underline{x}\|^2 \|\underline{y}\|^2 \geq \langle \underline{x},\underline{y}\rangle^2$

If $\underline{y} = \underline{0}$ CS becomes

$$\begin{array}{ccc} |\langle \underline{x},\underline{0}\rangle| & \leq & \|\underline{x}\|\cdot\|\underline{0}\| \\ \| & & \\ 0 & \leq & \|\underline{x}\|\cdot 0 \end{array}$$

(2) $\|\underline{x}+\underline{y}\|^2 = \langle \underline{x}+\underline{y}, \underline{x}+\underline{y} \rangle =$

$= \langle \underline{x}, \underline{x} \rangle + \langle \underline{x}, \underline{y} \rangle + \langle \underline{y}, \underline{x} \rangle + \langle \underline{y}, \underline{y} \rangle$

$= \|\underline{x}\|^2 + 2\langle \underline{x}, \underline{y} \rangle + \|\underline{y}\|^2$

CS

$\leq \|\underline{x}\|^2 + 2\|\underline{x}\|\|\underline{y}\| + \|\underline{y}\|^2 =$

$= (\|\underline{x}\| + \|\underline{y}\|)^2$

Def

A matrix $A \in \mathbb{R}^{n \times n}$ is called SYMMETRIC

if $A = A^T$

Example

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B \neq B^T$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Def

If $A = A^T \in \mathbb{R}^{n\times n}$, A is POSITIVE SEMIDEFINITE

if $\underline{x}^T A \underline{x} \geq 0$ for all $\underline{x} \in \mathbb{R}^n$

Moreover, if $\underline{x} \neq \underline{0} \Rightarrow \underline{x}^T A \underline{x} > 0$ then A is

POSITIVE DEFINITE

$$A = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

A is positive definite

$$\underline{x}^T A \underline{x} = 5x_1^2 + 2x_1x_2 + 4x_2^2 =$$

$$= 4x_1^2 + 3x_2^2 + (x_1 + x_2)^2 \geq 0$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \underline{y}^T B \underline{y} < 0 \qquad \underline{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -1 < 0$$

LEMMA

$\langle \cdot, \cdot \rangle$ is an inner product

$$\Updownarrow$$

there exists a symmetric positive definite (SPD) $A \in \mathbb{R}^{n \times n}$

s.t. $\langle \underline{x}, \underline{y} \rangle = \underline{y}^T A \underline{x}$

Example

$\langle \underline{x}, \underline{y} \rangle_A = \underline{y}^T \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \underline{x}$ is an inner product

Example

$\underline{x} \cdot \underline{y}$. Can we find a SPD matrix $M = I$

s.t. $\underline{x} \cdot \underline{y} = \underline{y}^T I \underline{x} = \underline{y}^T \underline{x}$

Proof

Suppose A is spd. $\langle \underline{x}, \underline{y} \rangle_A = \underline{y}^T A \underline{x}$

① $\underline{x}^T A \underline{x} \geq 0$ and $= 0 \Leftrightarrow \underline{x} = \underline{0}$

② $\langle \underline{x}, \underline{y} \rangle_A = \underline{y}^T A \underline{x} = (\underline{y}^T A \underline{x})^T = \underline{x}^T A^T \underline{y} = \underline{x}^T A \underline{y} = \langle \underline{y}, \underline{x} \rangle_A$

③ $\langle \alpha \underline{x} + \beta \underline{y}, \underline{z} \rangle_A = \underline{z}^T A (\alpha \underline{x} + \beta \underline{y}) = \alpha \underline{z}^T A \underline{x} + \beta \underline{z}^T A \underline{y} = \alpha \langle \underline{x}, \underline{z} \rangle_A + \beta \langle \underline{y}, \underline{z} \rangle_A$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \underline{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ component}$$

$\{e_1, \dots, \underline{e}_n\}$ is a basis

Then for all $\underline{x}, \underline{y} \in \mathbb{R}^n$ $\underline{x} = \sum_{i=1}^{n} \alpha_i \underline{e}_i$ and $\underline{y} = \sum_{j=1}^{n} \beta_j \underline{e}_j$

$$\langle \underline{x}, \underline{y} \rangle = \sum_{i,j=1}^{n} \alpha_i \beta_j \langle \underline{e}_i, \underline{e}_j \rangle$$

Define A s.t. $A_{ij} = \langle \underline{e}_i, \underline{e}_j \rangle$

Then $\langle \underline{x}, \underline{y} \rangle = \underline{y}^T A \underline{x}$ (exercise)

$\underline{z}^T A \underline{z} = \langle \underline{z}, \underline{z} \rangle \geq 0$ and $= 0$ iff $\underline{z} = 0$

Hence A is spd.

Suppose that $A \in \mathbb{R}^{m \times n}$

and let us e.g. a p-norm on $\mathbb{R}^n$ and $\mathbb{R}^m$

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \qquad \|\underline{x}\|_\infty = \max_i |x_i|$$

$$\|A\|_p = \max_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|_p}{\|\underline{x}\|_p}$$

Def

If $\|\cdot\|$ is a vector norm, the corresponding OPERATOR NORM is

$$\|A\| = \max_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|}{\|\underline{x}\|} = \max_{\|\underline{x}\|=1} \|A\underline{x}\|$$

LEMMA

Every operator norm satisfies

① $\|AB\| \leq \|A\| \cdot \|B\|$

② $\|A\underline{x}\| \leq \|A\| \, \|\underline{x}\|$