$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$ "EASY" TO COMPUTE

$$\|A\|_p = \max_{\|\underline{x}\|_p = 1} \|A\underline{x}\|_p$$ NP-HARD

<u>Exceptions</u> $\|A\|_1, \|A\|_2, \|A\|_\infty$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |A_{ij}|$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |A_{ij}|$$

$$\|A\|_1 = \max\{4, 6\} = 6$$

$$\|A\|_\infty = \max\{3, 7\} = 7$$

LEMMA

Let $A \in \mathbb{R}^{m \times n}$ and let λ be the largest eigenvalue of $A^T A$. Then, $\|A\|_2 = \sqrt{\lambda}$.

$$Bx = \lambda x, \quad x \neq 0$$

EIGENVALUE

EIGENVECTOR

A^TA is positive semidefinite

$$A^TAx = \lambda x$$

CLAIM

$\lambda \geq 0$

$$x^TA^TAx = \lambda x^Tx$$

$$\|Ax\|_2^2 = \lambda \|x\|^2 \Rightarrow \lambda = \frac{\|Ax\|^2}{\|x\|} \geq 0$$

$$B^T = (A^TA)^T = A^T(A^T)^T = A^TA = B$$

In the statement, $B = A^TA$

$$y^Ty = \sum_{i=1}^{n} y_i^2 = \|y\|_2^2$$

Proof

$$\|A\|_2 = \max_{\underline{x}\neq \underline{0}} \frac{\|A\underline{x}\|_2}{\|\underline{x}\|_2} \qquad \|A\|_2^2 = \max_{\underline{x}\neq \underline{0}} \frac{\|A\underline{x}\|_2^2}{\|\underline{x}\|_2^2} =$$

$$\max_{\underline{x}\neq \underline{0}} \frac{\underline{x}^T A^T A \underline{x}}{\underline{x}^T \underline{x}}.$$

By the "spectral theorem", every symmetric $B = Q\Lambda Q^T$ where $\Lambda \in \mathbb{R}^{n\times n}$ that contains the e'values and $Q \in \mathbb{R}^{n\times n}$ is orthogonal, $Q^TQ = I = QQ^T$

Hence, there are such Q, Λ s.t. $A^TA = Q\Lambda Q^T$

$$\|A\|_2^2 = \max_{\underline{x}\neq \underline{0}} \frac{\underline{x}^T Q\Lambda Q^T \underline{x}}{\underline{x}^T \underline{x}}$$

let $\underline{y} = Q^T\underline{x}$, then

$$\|A\|_2^2 = \max_{\underline{x}\neq\underline{0}} \frac{\underline{x}^T Q\Lambda Q^T\underline{x}}{\underline{x}^T QQ^T\underline{x}} = \max_{\underline{y}\neq\underline{0}} \frac{\underline{y}^T\Lambda\underline{y}}{\underline{y}^T\underline{y}} = \max_{\underline{y}\neq\underline{0}} \frac{\sum_{i=1}^n \lambda_i y_i^2}{\sum_{i=1}^n y_i^2}$$

$\lambda_i = \Lambda_{ii} =$ eigenvalues of A^TA

$$= \max_{\|\underline{y}\|_2 = 1} \sum_{i=1}^n \lambda_i y_i^2 \leq \lambda$$

and λ is obtainable by some choice of y_i!

Def

A function $\|\cdot\| : \mathbb{R}^{m \times n} \to \mathbb{R}$ is a NORM if

1. $\|A\| \geq 0$ and $\|A\| = 0 \iff A = 0$ for all $A \in \mathbb{R}^{m \times n}$
2. $\|A+B\| \leq \|A\| + \|B\|$ for all $A, B \in \mathbb{R}^{m \times n}$
3. $\|\alpha A\| = |\alpha| \, \|A\|$ for all $\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$

THEOREM

Every operator norm is a norm.

Proof

① $\|A\| = \max\limits_{\underline{x} \neq \underline{0}} \dfrac{\|A\underline{x}\|}{\|\underline{x}\|} \geq 0.$ Moreover, $\|A\| = 0 \iff A\underline{x} = \underline{0}$ for all $\underline{x}$

$\iff A = 0.$

③ $\|\alpha A\| = \max\limits_{\|x\|=1} \|\alpha A\underline{x}\| = \max\limits_{\|\underline{x}\|=1} |\alpha|\, \|A\underline{x}\| = |\alpha| \max\limits_{\|\underline{x}\|=1} \|A\underline{x}\| = |\alpha|\, \|A\|$

(2) $$\|A+B\| = \max_{\|\underline{x}\|=1} \|(A+B)\underline{x}\| = \max_{\|\underline{x}\|=1} \|A\underline{x}+B\underline{x}\|$$

$$\leq \max_{\|x\|=1} \left(\|A\underline{x}\| + \|B\underline{x}\|\right) \leq \max_{\|\underline{x}\|=1} \|A\underline{x}\| + \max_{\|\underline{x}\|=1} \|B\underline{x}\|$$

$$= \|A\| + \|B\|$$

Example of norms that are not operator norms

$$\|A\|_F^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} |A_{ij}|^2$$

$$\|A\|_{max} = \max_{i,j} |A_{ij}|$$

$$A\underline{x} = \underline{b} \qquad \underline{y} = \underline{x} + (\delta \underline{x})$$

$$(A + (\delta A))\underline{y} = (\underline{b} + (\delta \underline{b}))$$

$$(A + (\delta A))(\underline{x} + (\delta \underline{x})) = \underline{b} + (\delta \underline{b})$$

$$\cancel{A\underline{x}} + (\delta A)\underline{x} + A(\delta \underline{x}) + (\delta A)(\delta \underline{x}) = \cancel{\underline{b}} + \delta \underline{b}$$

$$(A + \delta A)(\delta \underline{x}) = \delta \underline{b} - (\delta A)\underline{x}$$

THEOREM

Let $\|\cdot\|$ be an operator norm. Then, if A is invertible and

$$\|\delta A\| < \frac{1}{\|A^{-1}\|}$$

then $A+\delta A$ is invertible.

$$N(A+\delta A) = \{\underline{0}\} \iff A+\delta A \text{ is invertible}$$

$$\frac{\|(A+\delta A)\underline{x}\|}{\|\underline{x}\|} \geq \frac{\|A\underline{x}\| - \|(\delta A)\underline{x}\|}{\|\underline{x}\|} = \frac{\|A\underline{x}\|}{\|\underline{x}\|} - \frac{\|(\delta A)\underline{x}\|}{\|\underline{x}\|}$$

$$\geq \min_{\underline{x} \neq \underline{0}} \frac{\|A\underline{x}\|}{\|\underline{x}\|} - \max_{\underline{x} \neq \underline{0}} \frac{\|(\delta A)\underline{x}\|}{\|\underline{x}\|}$$

$$A\underline{x} = \underline{y} \iff \underline{x} = A^{-1}\underline{y}$$

$$\min_{\underline{x}\neq\underline{0}} \frac{\|A\underline{x}\|}{\|\underline{x}\|} = \min_{\underline{y}\neq\underline{0}} \frac{\|\underline{y}\|}{\|A^{-1}\underline{y}\|} = \frac{1}{\max\limits_{\underline{y}\neq\underline{0}} \frac{\|A^{-1}\underline{y}\|}{\|\underline{y}\|}} = \frac{1}{\|A^{-1}\|}$$

$$\frac{\|(A+\delta A)\underline{x}\|}{\|\underline{x}\|} \geq \frac{1}{\|A^{-1}\|} - \|\delta A\| > 0$$

$\Rightarrow$ If $\underline{x}\neq\underline{0}$ then $(A+\delta A)\underline{x} \neq \underline{0} \Rightarrow N(A+\delta A) = \{\underline{0}\}$

COROLLARY

Under the same assumptions,

$$\|(A+\delta A)^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|\delta A\|\,\|A^{-1}\|}$$

Proof

$$(XY)^{-1} = Y^{-1}X^{-1}$$

$$(I+(\delta A)A^{-1})A = A + \delta A \cdot A^{-1}A = A + \delta A$$

$$\|(A+\delta A)^{-1}\| = \| A^{-1}(I+(\delta A)A^{-1})^{-1} \|$$

$$\leq \|A^{-1}\| \cdot \|(I+(\delta A)A^{-1})^{-1}\|$$

$$\max_{x \neq 0} \frac{\|(I+(\delta A)A^{-1})^{-1}x\|}{\|x\|} = \max_{y \neq 0} \frac{\|y\|}{\|y + \delta A A^{-1} y\|}$$

$$(I+\delta A \cdot A^{-1})^{-1}x = y$$

$$x = (I+\delta A \cdot A^{-1})y$$

$$\|y + \delta A A^{-1} y\| \geq \|y\| - \|\delta A \cdot A^{-1} y\|$$

$$\geq \|y\| - \|\delta A\| \cdot \|A^{-1}\| \cdot \|y\|$$

$$= \|y\| \left(1 - \|\delta A\| \cdot \|A^{-1}\|\right)$$

$$\|(I+(\delta A)A^{-1})^{-1}\| \leq \max_{y\neq 0} \frac{\|y\|}{\|y\|(1-\|\delta A\|\,\|A^{-1}\|)} =$$

$$= \frac{1}{1-\|\delta A\|\cdot\|A^{-1}\|}$$