Def

If $\|\cdot\|$ is a matrix norm and if $A \in \mathbb{R}^{n\times n}$ is invertible, then the CONDITION NUMBER (wrt $\|\cdot\|$) of A is

$$\kappa(A) = \|A\| \, \|A^{-1}\|$$

THEOREM

Under the same assumptions (as last week), suppose

$$A\underline{x} = \underline{b} \quad \text{and} \quad (A+\delta A)\underline{y} = (\underline{b}+\underline{\delta b}).$$

Then,

$$\frac{\|\underline{y}-\underline{x}\|}{\|\underline{x}\|} \leq \frac{K(A)}{1-\frac{\|\delta A\|}{\|A\|}K(A)}\left(\frac{\|\underline{\delta b}\|}{\|\underline{b}\|}+\frac{\|\delta A\|}{\|A\|}\right)$$

$$A\underline{x} = \underline{b} \qquad (A+\delta A)y = \underline{b} + \underline{\delta b}$$

$$(A+\delta A)(y - \underline{x}) = \underline{\delta b} - (\delta A)\underline{x} \qquad y - \underline{x} = (A+\delta A)^{-1}(\underline{\delta b} - (\delta A)\underline{x})$$

$$\|y - \underline{x}\| \le \|(A+\delta A)^{-1}\| \, \|\underline{\delta b} - (\delta A)\underline{x}\| \le \|(A+\delta A)^{-1}\| \left(\|\underline{\delta b}\| + \|(\delta A)\underline{x}\|\right)$$

$$\le \frac{\|A^{-1}\|}{1 - \|\delta A\| \, \|A^{-1}\|} \left(\|\underline{\delta b}\| + \|(\delta A)\underline{x}\|\right)$$

NOTE THAT

$$\|(\delta A)\underline{x}\| \le \|\delta A\| \cdot \|\underline{x}\|$$

$$\frac{\|\underline{y}-\underline{x}\|}{\|\underline{x}\|} \leq \frac{\|A^{-1}\|\,\|A\|}{1-\frac{\|\delta A\|}{\|A\|}\|A^{-1}\|\,\|A\|}\left(\frac{\|\delta \underline{b}\|}{\|A\|\,\|\underline{x}\|}+\frac{\|\delta A\|}{\|A\|}\right)=$$

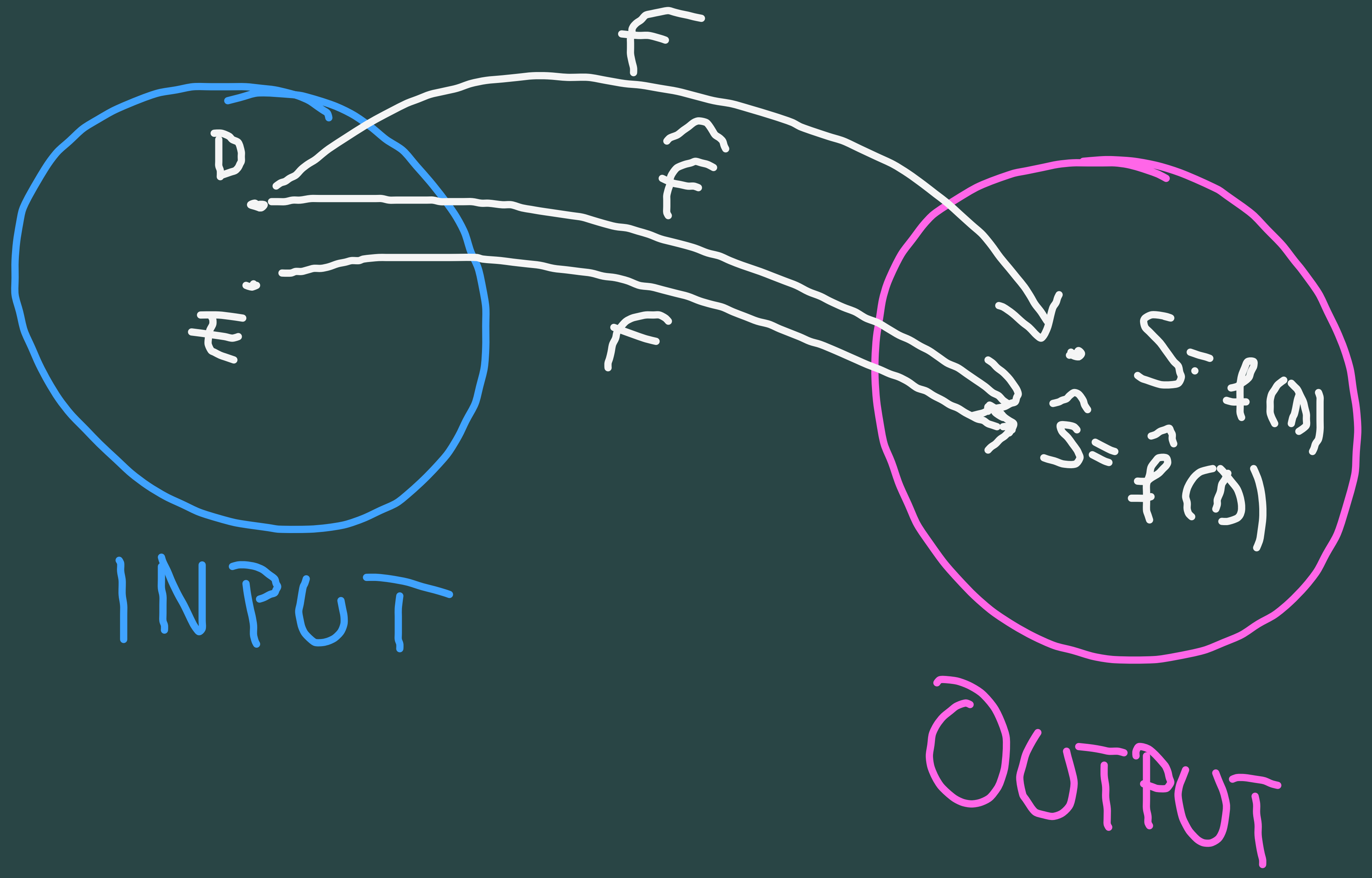
$$= \frac{k(A)}{1-\frac{\|\delta A\|}{\|A\|}k(A)}\left(\frac{\|\delta \underline{b}\|}{\|A\|\,\|x\|}+\frac{\|\delta A\|}{\|A\|}\right)$$

TRICK:

$\underline{b}=A\underline{x} \Rightarrow \|\underline{b}\| \leq \|A\|\,\|\underline{x}\|$

$\Rightarrow \frac{1}{\|A\|\,\|\underline{x}\|} \leq \frac{1}{\|\underline{b}\|}$

$$\leq \frac{k(A)}{1-\frac{\|\delta A\|}{\|A\|}k(A)}\left(\frac{\|\delta \underline{b}\|}{\|\underline{b}\|}+\frac{\|\delta A\|}{\|A\|}\right)$$


D
E
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F̂
F
S = f(D)
Ŝ = f̂(D)
INPUT
OUTPUT

$$A\underline{x} = \underline{b}$$

$$\begin{aligned} x_1 + 1.001\,x_2 &= 1 \\ x_1 + x_2 &= 1 \end{aligned} \Longrightarrow \begin{aligned} x_1 &= 1 \\ x_2 &= 0 \end{aligned}$$

$$a+ib \qquad i^2=-1$$

$$|a+ib|=\sqrt{a^2+b^2}$$

$$\overline{a+ib}=a-ib$$

$$\left\|\begin{bmatrix}2+3i\\ i\end{bmatrix}\right\|_2 = \left(|2+3i|^2+|i|^2\right)^{\frac{1}{2}} = \sqrt{14}$$

$$\langle \underline{x}, \underline{y} \rangle = \overline{\langle \underline{y}, \underline{x} \rangle} \quad \forall\, \underline{x}, \underline{y} \in \mathbb{C}^n$$

$$\langle \alpha \underline{x}, \underline{y} \rangle = \alpha \langle \underline{x}, \underline{y} \rangle \quad \forall\, \alpha \in \mathbb{C}\ \forall\, \underline{x}, \underline{y} \in \mathbb{C}^n$$

$$\langle \underline{x}, \beta \underline{y} \rangle = \overline{\beta} \langle \underline{x}, \underline{y} \rangle$$

$$A^* = \overline{\left(A^T\right)} = \left(\overline{A}\right)^T$$

$$A = \begin{bmatrix} 2+i & 3i \\ 4-i & 5+6i \end{bmatrix} \qquad A^T = \begin{bmatrix} 2+i & 4-i \\ 3i & 5+6i \end{bmatrix}$$

$$A^* = \begin{bmatrix} 2-i & 4+i \\ -3i & 5-6i \end{bmatrix}$$

If $A = A^*$, A is called HERMITIAN

Example $A = \begin{bmatrix} 1 & 3+i \\ 3-i & 2 \end{bmatrix}$

$$A\underline{x} = \lambda \underline{x}$$

$$A \in \mathbb{C}^{n \times n}$$

$$\underline{x} \in \mathbb{C}^{n}$$

$$\lambda \in \mathbb{C}$$

$$\boxed{\underline{x} \neq \underline{0}}$$

IMPORTANT

$$(A-\lambda I)\underline{x}=\underline{0}, \quad \underline{x}\neq\underline{0}$$

$$\underline{x}\in N(A-\lambda I), \quad \underline{x}\neq\underline{0}$$

λ is an eigenvalue of A iff $N(A-\lambda I)\neq\{\underline{0}\}$

and, in this case, any nonzero element of $N(A-\lambda I)$ is an eigenvector

$E_\lambda = N(A - \lambda I)$ is called the EIGENSPACE of A associated with λ.

$\mu_G(\lambda) = \dim N(A - \lambda I)$ is called the GEOMETRIC MULTIPLICITY of λ.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad N(A - 1 \cdot I) = N\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$\begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

1 is an e'value of A with geom. mult. 1

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad N(A - 1 \cdot I) = N\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

1 is an e-value of A with geom. mult. is 2

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\underline{1}$ is an e'value of A with geom. mult. ? (EXERCISE)

$$N(A-\lambda I) \neq \{\underline{0}\} \iff \det(A-\lambda I) = 0$$

<u>Def</u>

Given $A \in \mathbb{C}^{n \times n}$, the CHARACTRESTIC POLYNOMIAL of A

is $p_A(\lambda) = \det(A - \lambda I)$

$M \in \mathbb{C}^{n \times n}$

$$\det M = \sum_{i=1}^{n} (-1)^{i+j} M_{ij} \det [M]_{\hat{i}\hat{j}}$$

LAPLACE EXPANSION

a matrix obtained by deleting row i and column j

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \det A = \sum_{i=1}^{3} (-1)^{i+1} A_{i1} \det [A]_{i1}$$

$$= 1 \cdot 1 \cdot \det \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 4 \cdot \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 7 \cdot \det \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$$

$$= 1\,(45-48) - 4\,(18-24) + 7\,(12-15) =$$

$$= -3 + 24 - 21 = 0$$

LEMMA

$p_A(\lambda)$, where $A \in \mathbb{C}^{n \times n}$, is a polynomial of degree n with leading coefficient is $(-1)^n$.

Proof

BASE CASE $n=1$

$$\det(a - 1 \cdot \lambda) = a - \lambda$$

INDUCTION $(n-1 \longrightarrow n)$

Claim: $\det(B + C\lambda)$ has degree $\leq n$ where $B, C \in \mathbb{C}^{n \times n}$

(proof by Laplace expansion)

$$\det(A-\lambda I) = \sum_{i=1}^{n} (-1)^{i+j} (A-\lambda I)_{i1} \det\left[(A-\lambda I)_{i1}\right]$$

$$= \underbrace{(-1)^{2} \underbrace{(A-\lambda I)_{11}}_{\text{degree } 1} \det\underbrace{\left((A-\lambda I)_{11}\right)}_{\text{degree} = n-1}}_{\text{degree} = n} + \underbrace{\sum_{i=2}^{n} (-1)^{i+j} \cdot \underbrace{(A-\lambda I)_{i1}}_{\text{degree} = 0} \underbrace{\det \ldots}_{\text{degree} \leq n-1}}_{\text{degree} \leq n-1}$$