1

$$p_A(\lambda) = \sum_{i=0}^{n} a_i \lambda^i = a_n \prod_{i=1}^{n} (\lambda - \lambda_i) = a_n \prod_{\sum m_i = n} (\lambda - \mu_i)^{m_i}$$

$(a_n \neq 0)$ (λ_i can be repeated) (μ_i are distinct)

m_i is the MULTIPLICITY of the root μ_i

Example

$$\lambda^4 - 5\lambda^3 + \lambda^2 + 0\cdot\lambda + 0\cdot\lambda^0 = 1\cdot(\lambda-0)(\lambda-0)(\lambda-2)(\lambda-3) = 1\cdot(\lambda-0)^2\cdot(\lambda-2)\cdot(\lambda-3)$$

Def

If $A \in \mathbb{C}^{n\times n}$, the algebraic multiplicity $\mu_A(\lambda)$ of an eigenvalue λ is the multiplicity of λ as a root of $p_A(\lambda)$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$$

$$p_A(\lambda) = \lambda^2$$

$$\mu_A(0) = 2$$

$$A - 0 \cdot I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \dim N(A - 0 \cdot I) = 2$$

$$\mu_G(0) = 2$$

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

$$p_B(\lambda) = \lambda^2$$

$$\mu_A(0) = 2$$

$$N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\parallel$$

$$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\mu_G(0) = 1$$

Fact: $\mu_G(\lambda) \leq \mu_A(\lambda)$

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \quad p_A(\lambda) = \det\left(\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix}\right) =$$

$$= (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$$

$$N(A-2I) = N\left(\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 2x_2 \\ x_1 - x_2 \end{bmatrix}$$

THEOREM

$(k \leq n)$

Let $A \in \mathbb{C}^{n \times n}$ and suppose that $\lambda_1, \dots, \lambda_k$ are distinct e'values with e'vectors $\underline{v}_1, \dots, \underline{v}_k$.

Then $\underline{v}_1, \dots, \underline{v}_k$ are linearly independent.

GOAL

If $\sum_{i=1}^{k} \alpha_i \underline{v}_i = \underline{0}$ then $\alpha_1 = \dots = \alpha_k = 0$

$\boxed{k=1}$

$A\underline{v}_1 = \lambda_1 \underline{v}_1 \, , \; \underline{v}_1 \neq \underline{0}$

$\alpha_1 \underline{v}_1 = \underline{0} \Longrightarrow \alpha_1 = 0$

(because $\underline{v}_1 \neq \underline{0}$)

$\boxed{k \to k+1}$ Wlog $\lambda_{k+1} \neq 0$

$$\sum_{i=1}^{k+1} \alpha_i \underline{v}_i = \underline{0} \implies A\left(\sum \alpha_i v_i\right) = A\,\underline{0}$$

$$\sum \alpha_i A \underline{v}_i = \sum \alpha_i \lambda_i \underline{v}_i = \underline{0}$$

$$\sum_{i=1}^{k} \alpha_i \lambda_i \underline{v}_i + \alpha_{k+1} \lambda_{k+1} \underline{v}_{k+1} = \underline{0}$$

$$\alpha_{k+1} \underline{v}_{k+1} = -\sum_{i=1}^{k} \alpha_i \frac{\lambda_i}{\lambda_{k+1}} \underline{v}_i$$

$$\underline{0} = \sum_{i=1}^{k} \alpha_i \underline{v}_i + \alpha_{k+1} \underline{v}_{k+1} = \sum_{i=1}^{k} \left(\alpha_i \underline{v}_i - \frac{\alpha_i \lambda_i}{\lambda_{k+1}} \underline{v}_i \right)$$

$$= \sum_{i=1}^{k} \alpha_i \left(1 - \frac{\lambda_i}{\lambda_{k+1}}\right) \underline{v}_i \Rightarrow \alpha_i \underbrace{\left(1 - \frac{\lambda_i}{\lambda_{k+1}}\right)}_{\neq 0 \text{ because } \lambda_i \neq \lambda_{k+1}} = 0 \text{ for all } i = 1, \ldots, k$$

$$\Rightarrow \alpha_i = 0 \text{ for } i = 1, \ldots, k$$

$$\Rightarrow \underline{0} = \alpha_{k+1} \underline{v}_{k+1} \Rightarrow \alpha_{k+1} = 0.$$

If $A \in \mathbb{C}^{n \times n}$ and has all distinct e'vals then there is a basis $\{\underline{v}_1, \dots, \underline{v}_n\}$ of $\mathbb{C}^n$ made by e'vecs of A.

$$V = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \end{bmatrix}$$

$$AV = \begin{bmatrix} A\underline{v}_1 & \cdots & A\underline{v}_n \end{bmatrix} = \begin{bmatrix} \underline{v}_1\lambda_1 & \cdots & \underline{v}_n\lambda_n \end{bmatrix}$$

$$= V \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = V\Lambda$$

$$AV = V\Lambda$$

matrix whose col's are eigenvectors

DIAGONAL matrix whose diagonal elements are the e'vals

$$A = V\Lambda V^{-1}$$

$$V^{-1}AV = \Lambda$$

A is DIAGONALIZABLE

Example

Every matrix with distinct eigenvalues is diagonalizable

Non-example

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Def

If $A = A^* \in \mathbb{C}^{n \times n}$ then A is HERMITIAN

(Special case: $A \in \mathbb{R}^{n \times n}$ and SYMMETRIC).

If $U \in \mathbb{C}^{n \times n}$ s.t. $U^* U = I = U U^*$ then U is UNITARY.

(Special case: $U \in \mathbb{R}^{n \times n}$ and $U^T U = U U^T = I$, U is ORTHOGONAL)

If $U = \begin{bmatrix} \underline{u}_1, \ldots, \underline{u}_n \end{bmatrix}$ is unitary, then

$$(U^* U)_{ij} = \underline{u}_i^* \underline{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The columns of a unitary matrix are orthonormal wrt dot product

$B = B^* \Rightarrow \exists$ Q unitary

$\Lambda \in \mathbb{R}^{n \times n}$, diagonal

s.t. $B = Q \Lambda Q^*$

$\Updownarrow$

$Q^* B Q = \Lambda$ (B is UNITARILY DIAGONALIZABLE)

$\|$

$Q^{-1} B Q$

LEMMA

If $B = B^*$, then the e'vals of B are real numbers.

Proof

λ is an e'value $\Rightarrow B\underline{v} = \lambda \underline{v}$, $\underline{v} \neq \underline{0}$

Then. $\underline{v}^* B \underline{v} = \lambda \underline{v}^* \underline{v} = \lambda \|\underline{v}\|_2^2 \Rightarrow \lambda = \dfrac{\underline{v}^* B \underline{v}}{\|\underline{v}\|_2^2} \in \mathbb{R}$

$(\underline{v}^* B \underline{v})^* = \underline{v}^* B^* \underline{v} = \underline{v}^* B \underline{v} \in \mathbb{R}$

LEMMA

Let $B = B^*$, and let $\lambda_1 \neq \lambda_2$ be distinct e-values of B with e-vecs q_1, q_2. Then $q_1^* q_2 = 0$.

Proof

$Bq_2 = \lambda_2 q_2 \qquad q_1^* B = \lambda_1 q_1^*$

$Bq_1 = \lambda_1 q_1$

$$\left.\begin{aligned} \lambda_2 q_1^* q_2 &= q_1^* B q_2 \\ \lambda_1 q_1^* q_2 &= q_1^* B q_2 \end{aligned}\right\} \Longrightarrow \lambda_1 q_1^* q_2 = \lambda_2 q_1^* q_2 \Leftrightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{q_1^* q_2}_{= 0} = 0$$

THEOREM (SPECTRAL THEOREM)

If $B = B^*$ there are Q unitary and Λ real diagonal s.t. $Q^* B Q = \Lambda$

Proof (INDUCTION ON THE SIZE n OF $B \in \mathbb{C}^{n\times n}$)

$\boxed{n=1}$ $B = B^* \in \mathbb{C}^{1\times 1} \iff B \in \mathbb{R}$

Find $\Lambda \in \mathbb{R}$, $Q \in \mathbb{C}$ s.t. $Q^*Q = 1$ $\quad Q = \rho e^{i\theta} = \rho(\cos\theta + \sin\theta \cdot i)$

$\Lambda = B$ $Q = 1$ works $\quad Q^* B Q = 1 \cdot B \cdot 1 = B$

$Q^*Q = 1 \cdot 1 = 1$

$$\boxed{n \to n+1}$$

$$B = B^* \in \mathbb{C}^{(n+1)\times(n+1)}$$

$Bq_1 = \lambda_1 q_1$, $q_1 \neq \underline{0}$ WLOG, $\|q_1\|_2^2 = q_1^* q_1 = 1$

Claim: $\exists\, q_2, \cdots, q_{n+1} \in \mathbb{C}^{n+1}$ s.t. $\{q_1, q_2, \cdots, q_{n+1}\}$ is ORTHONORMAL wrt dot product

$$\tilde{Q} = [q_1, q_2, \cdots, q_{n+1}]$$ is UNITARY

$$\tilde{Q}^* B \tilde{Q} = \begin{bmatrix} q_1^* \\ U^* \end{bmatrix} B \begin{bmatrix} q_1 & U \end{bmatrix} = \begin{bmatrix} q_1^* B q_1 & q_1^* B U \\ U^* B q_1 & U^* B U \end{bmatrix}$$

$$U = [q_2, \dots, q_{n+1}]$$

$$= \begin{bmatrix} \lambda_1 q_1^* q_1 & \lambda_1 q_1^* U \\ \lambda_1 U^* q_1 & U^* B U \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & U^* B U \end{bmatrix}$$

$\to$ Hermitian $(U^*BU)^* = U^*BU$

$\in \mathbb{C}^{n \times n}$

By assumption, $\exists\ \tilde{U}$ unitary s.t. $\tilde{U}^*(U^*BU)\tilde{U} = \tilde{\Lambda} \in \mathbb{R}^{n \times n}$

diagonal

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \tilde{U}^* \end{bmatrix} \tilde{Q}^*}_{=Q^*} B \underbrace{\tilde{Q} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{U} \end{bmatrix}}_{=Q} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{U}^* \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & U^*BU \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \tilde{U}^* U^* B U \tilde{U} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \tilde{\Lambda} \end{bmatrix} =: \Lambda$$

real diagonal

<u>Claim 1</u> if U_1, U_2 unitary, then $U_1 U_2$ is.

$$(U_1 U_2)^* U_1 U_2 = U_2^* U_1^* U_1 U_2 = U_2^* I U_2$$
$$= U_2^* U_2 = I$$

<u>Claim 2</u> If U is unitary, $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$ is:

$$\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & U^* \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_n \end{bmatrix} = I_{n+1}$$