1

Def $A \in \mathbb{C}^{n \times n}$ is Hermitian if $A = A^*$

$\in \mathbb{R}^{n \times n}$ is symmetric if $A = A^T$

<u>Theorem</u>

If $A = A^*$ then there exists a decomposition

$$A = Q \Lambda Q^*$$ where $\Lambda \in \mathbb{R}^{n \times n}$ and diag

$QQ^* = I$ (unitary)

For a Hermitian matrix we can characterize the eigenvalues via the "Rayleigh quotient

**Def:** $R(A,x) := \dfrac{x^* A x}{x^* x}$ $\quad x \neq 0$ (also possible $x^* A x$ for $\|x\|=1$

Property $R(A,x) \in \mathbb{R}$

Proof $\dfrac{x^* A x}{\|x\|_2^2} = \dfrac{x^* A^* x}{\|x\|_2^2} = \dfrac{(x^* A x)^*}{\|x\|_2^2} = \dfrac{\overline{(x^* A x)}}{\|x\|_2^2} \Rightarrow \mathbb{R}$

I want to prove that the eigenvalues of $A$ satisfy

(order the eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ )

$$\lambda_1 \leq R(A,x) \leq \lambda_n$$

$$\lambda_1 = \min_{x \neq 0} R(A,x), \quad \lambda_n = \max_{x \neq 0} R(A,x)$$

---

Let us expand $x = \sum_{i=1}^{n} \alpha_i q_i = Q\alpha \qquad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

then it follows that $x^* A x = \sum |\alpha_i|^2 \lambda_i, \quad x^* x = \sum |\alpha_i|^2$

$$\lambda_1 = \frac{\sum |\alpha_i|^2 \lambda_1}{\sum |\alpha_i|^2} \leq R(A,x) = \frac{\sum |\alpha_i|^2 \lambda_i}{\sum |\alpha_j|^2} \leq \frac{\sum |\alpha_i|^2 \lambda_n}{\sum |\alpha_i|} = \lambda_n$$

Those extrema are attained for

$$\alpha = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad x = Q \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \qquad \Downarrow \quad \alpha = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$q_1$ $\qquad$ $q_n$

---

Courant Fisher

$$\boxed{\lambda_k = \min_{\substack{U_k \\ \text{of} \\ \text{dimension } k}} \ \max_{0 \neq x \in U_k} R(A,x)}$$

# Similarity transformation

$$A = S B S^{-1} \qquad \det S \neq 0$$

B should hopefully be "simpler" than A

Similarity implies that A and B are represent the same mapping in a different coord. system

$y = Ax$ , express $x = S\alpha$ , $y = S\beta$ then

$$S\beta = A S \alpha \Rightarrow \beta = S^{-1} A S \alpha = B\alpha$$

This can be used for powers of A

$$A = SBS^{-1} \Longrightarrow A^n = S B^n S^{-1}$$

$$\sum \alpha_i A^i \Longrightarrow \sum \alpha_i A^i = S\left(\sum B^i \alpha_i\right) S^{-1}$$

$$p(A) = S(p(B)) S^{-1}$$

$$f(A) = S f(B) S^{-1}$$ (f should have a convergent Taylor exp.

Lemma If $A$ and $B$ are similar $\left( A = SBS^{-1} \right)$

then the associated char. poly. are equal

$$P_A(\lambda) := \det(A - \lambda I) = \det(SBS^{-1} - \lambda SS^{-1}) = \det(S(B - \lambda I)S^{-1})$$

$$= \det S \cdot \det(B - \lambda I) \cdot \det S^{-1} = \det(B - \lambda I) = P_B(\lambda)$$

# Schur form (with special thanks to FRANCIS)

<u>Theorem</u> for $A \in \mathbb{C}^{n \times n}$ there exists a a unitary $Q \in \mathbb{C}^{n \times n}$ such that $A = QTQ^*$ where $T$ is upper triangular. $T$ is called the Schur form of $A$

---

Proof by induction on the size of $A$

For $k=1$ $\quad A=\alpha \quad Q=1$

Assume the Th. holds for $k$ $\Rightarrow$ prove it for $k+1$

$A \in \mathbb{C}^{k+1 \times k+1}, \quad A q_1 = \lambda_1 q_1, \quad \|q_1\|_2 = 1$

Define $[q_2, \dots, q_{k+1}] =: \tilde{Q} \quad [q_1, \dots, q_{k+1}] =: Q$ unitary

Then construct $Q^* A Q = [q_1 | \tilde{Q}]^* A [q_1 | \tilde{Q}]$

$$Q^* A Q = \begin{bmatrix} q_1^* A q_1 & q_1^* A \tilde{Q} \\ \tilde{Q}^* A q_1 & \tilde{Q}^* A \tilde{Q} \end{bmatrix} = \left[\begin{array}{c|c} \lambda_1 q_1^* q_1 & q_1^* A \tilde{Q} \\ \hline \lambda_1 \underbrace{\tilde{Q}^* q_1}_{=0} & \tilde{Q}^* A \tilde{Q} \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & \tilde{Q}^* A \tilde{Q} \end{array}\right] \to k \times k$$

By induction hypothesis

$$\tilde{Q}^* A \tilde{Q} = Q_1 T_1 Q_1^* \qquad T_1 \text{ triang}$$

$$\underbrace{\left[\begin{array}{c|c} 1 & \\ \hline & Q_1 \end{array}\right]}_{\text{unitary}} Q^* A Q \underbrace{\left[\begin{array}{c|c} 1 & \\ \hline & Q_1^* \end{array}\right]}_{\text{unitary}} = \begin{bmatrix} \lambda_1 & * \\ 0 & T_1 \end{bmatrix}$$

# Jordan normal form important theoretical tool

**Theorem** any matrix $A \in \mathbb{C}^{n \times n}$ has a decomposition

$$A = P J P^{-1} \qquad \det P \neq 0$$

and $J = \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_\ell}(\lambda_\ell) \end{bmatrix}$ and $J_{n_i}(\lambda_i) = \underbrace{\begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}}_{n_i}$

## Example

$$J_3(\alpha) = \begin{pmatrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{pmatrix}$$

eigenvalue $\alpha$ has alg. mult. $3 = n_i$

$\alpha$ has geometric multiplicity $1$

Null space of $\left(J_3(\alpha) - \alpha I\right) = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}$ is $\mathrm{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

## Example 2

$$S^{-1} A S = \left[\begin{array}{c|cc|cccc|ccc}
\alpha & & & & & & & & & \\
\hline
 & \alpha & 1 & & & & & & & \\
 & & \alpha & & & & & & & \\
\hline
 & & & \alpha & 1 & & & & & \\
 & & & & \alpha & 1 & & & & \\
 & & & & & \alpha & 1 & & & \\
 & & & & & & \alpha & & & \\
\hline
 & & & & & & & \beta & 1 & \\
 & & & & & & & & \beta & 1 \\
 & & & & & & & & & \beta
\end{array}\right]$$

<u>alg mult.</u> is 7 for $\alpha$

<u>geom m</u> is 3 for $\alpha$

3 for $\beta$

1 for $\beta$

$\{Se_1, Se_2, Se_4\}$ eigv of $\alpha$

$\{Se_8\}$ is an eigv of $\beta$

The other cols of $S$ are <u>generalized eigenvectors</u>

Detail example

$$S^{-1}AS = \begin{bmatrix} \alpha & 1 & \\ & \alpha & 1 \\ & & \alpha \end{bmatrix}$$

$$S^{-1}(A-\alpha I)S = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

$$S^{-1}(A-\alpha I)^2 S = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

$$S^{-1}(A-\alpha I)^3 S = \begin{bmatrix} & 0 & \end{bmatrix}$$

$Se_1$ is an eigenvector $(A-\alpha I)Se_1 = 0$