The matrix exponential and its use in systems of Diff Eq

The scalar function $\quad e^t = \sum_{j=0}^{\infty} \frac{t^j}{j!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots$

uniform convergence for $t \in$ bounded set in $\mathbb{C}$

$$\frac{d}{dt} e^t = \sum_{i=0}^{\infty} \frac{d}{dt} \frac{t^j}{j!} = \sum_{j=1}^{\infty} j \frac{t^{j-1}}{j!} = \sum_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!} = \sum_{j=0}^{\infty} \frac{t^j}{j!} = e^t$$

$$\frac{d}{dt} e^{at} = \quad \cdots = a e^{at} \qquad a \in \mathbb{C}$$

Therefore

$$\begin{cases} x'(t) = a\,x(t) & (-)_0 \\ x(0) = x_0 \in \mathbb{C} & \longrightarrow \text{initial cond.} \end{cases}$$

has the solution $x(t) = e^{at} x_0$

Can we extend to systems of diff. eq.

$$\begin{cases} x'(t) = A\,x(t) & x(t) \in \mathbb{C}^n \\ x(0) = x_0 & x_0 \in \mathbb{C}^n \end{cases}$$

Formally define $(A \in \mathbb{C}^{n \times n})$

$$e^{tA} = \sum_{j=0}^{\infty} \frac{t^j A^j}{j!} = I + tA + \frac{t^2A^2}{2} + \frac{t^3A^3}{6} + \cdots$$

This is a Cauchy sequence if $S_N = \sum_{j=0}^{N} \frac{t^j A^j}{j!}$ satisfies

$$\| S_N(t) - S_M(t) \| \to 0 \quad \text{if } N, M \to \infty$$

Matrices form a normed vector space since we have a basis

$$A = \sum a_{ij} \, e_i e_j^T = \sum a_{ij} \begin{array}{cc} & j \\ \left[\begin{array}{ccc} & \vdots & \\ \cdots & 1 & \\ & & \end{array} \right] & i \end{array}$$

$$\|S_N - S_M\| = \left\|\sum_{M+1}^{N} \frac{t^j A^j}{j!}\right\| \underset{\uparrow \text{T.I.}}{\leq} \sum_{M+1}^{N} \left\|\frac{t^j A^j}{j!}\right\| \leq \sum_{M+1}^{N} \frac{|t|^j \|A\|^j}{j!}$$

$M < N$

$$\leq \sum_{k=0}^{N-(M+1)} \frac{(|t|\|A\|)^{k+M+1}}{(k+(M+1))!} \leq \frac{(|t|\|A\|)^{M+1}}{(M+1)!} \underbrace{\sum_{k=0}^{N-(M+1)} \frac{(|t|\|A\|)^k}{k!}} \leq \underbrace{\frac{(|t|\|A\|)^{M+1}}{(M+1)!}}_{\to 0} \underbrace{e^{|t|\|A\|}}_{\text{bounded}}$$

this is a Cauchy sequence

Systems of diff. eq.

$x(t) \in \mathbb{C}^n$

$$x'(t) = A\,x(t) \qquad t > 0$$

$$x(0) = x_0 \in \mathbb{C}^n$$

Derive $\frac{de^{At}}{dt}$

$$= \frac{d}{dt}\left[I + tA + \frac{t^2A^2}{2} + \dots\right] = \sum_0^\infty \frac{d}{dt}\frac{t^jA^j}{j!} = \sum_{j=1}^\infty \frac{jt^{j-1}}{j!}A^j$$

$$= \sum_{j=1}^\infty A \cdot \frac{A^{j-1}}{(j-1)!} = Ae^{At}$$

Initial conditions

$$\left(e^{At}x_0\right)(t=0) = x_0$$

Let us suppose A is diagonalizable

$$A = X \Lambda X^{-1} \Rightarrow A^n = X \Lambda^n X^{-1}$$

$$e^{A} = X e^{\Lambda} X^{-1}, \quad e^{At} = X e^{\Lambda t} X^{-1} \text{ and also } e^{At} x_0 = X e^{\Lambda t} \underbrace{X^{-1} x_0}_{\hat{x}_0}$$

Solution of system of D.E. $x(t) = e^{At} x_0 = X e^{\Lambda t} \hat{x}_0$

Verify that $\begin{cases} x'(t) = A x(t) \\ x(0) = x_0 \end{cases}$

$$= X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \hat{x}_0$$

Example

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & \\ & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$e^{At} x_0 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \frac{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}{2} x_0 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

What about non-diagonalable case?

Use the Jordan normal form

$$A = PJP^{-1} \Rightarrow \text{where } J = \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_\ell}(\lambda_\ell) \end{bmatrix}$$

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

geom mult 1

alg mult n_i

$$e^{Jt} = \begin{bmatrix} e^{J_{n_1}(\lambda_1)t} & & \\ & \ddots & \\ & & e^{J_{n_\ell}(\lambda_\ell)t} \end{bmatrix} = \begin{bmatrix} \nabla & & \\ & \ddots & \\ & & \nabla \end{bmatrix} \quad e^{J_{n_i}(\lambda_i)}$$

$$e^{J_n(\lambda)t} = e^{\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} t} = \begin{bmatrix} e^{\lambda t} & & \\ & \ddots & \\ & & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \vdots \\ & \ddots & \ddots & \ddots & \frac{t^2}{2} \\ & & \ddots & \ddots & t \\ & & & & 1 \end{bmatrix}$$

no proof

expansion of e^{Nt}

Analysis of (nonlinear) diff. eqs.

$$x'(t) = F(x(t)) \quad \text{for } t > 0 \quad \text{and } x(0) = x_0 \in \mathbb{C}^n$$

$$x(t) \quad \mathbb{R} \to \mathbb{R}^n$$

$$F(x(t)) \quad \mathbb{R}^n \to \mathbb{R}^n$$

<u>Lotka-Volterra</u>

$$F(x) = \begin{pmatrix} \frac{2}{3} x_1 - \frac{4}{3} x_1 x_2 \\ \\ x_1 x_2 - x_2 \end{pmatrix}$$

Equilibrium point $\tilde{x}$

$$F(\tilde{x}) = 0 \quad \tilde{x}_1 = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix} \quad \tilde{x}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Linearize around equilibrium points

$$DF(x) = \begin{bmatrix} \frac{\partial F_1(x)}{\partial x_1} & \frac{\partial F_1(x)}{\partial x_2} \\ \frac{\partial F_2(x)}{\partial x_2} & \frac{\partial F_2(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} - \frac{4}{3}x_2 & -\frac{4}{3}x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$$

$$F(x) \cong \underbrace{F(\tilde{x})}_{\substack{\| \\ 0}} + DF(\tilde{x})(x - \tilde{x})$$

$0 \to$ at equil pt

$$\frac{d}{dt}(\underbrace{x(t) - \tilde{x}}) = x'(t) = DF(\tilde{x})(x - \tilde{x})$$

$$y'(t) = \underbrace{DF(\tilde{x})}_{A}\, y(t)$$

where $DF(\tilde{x}_1) = \begin{bmatrix} 0 & -\frac{4}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$ $\quad \lambda^2 + \frac{2}{3}$

$\lambda_1 = \pm i\sqrt{\frac{2}{3}}$

$DF(\tilde{x}_2) = \begin{bmatrix} 2/3 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda_1 = 2/3$, $\lambda_2 = -1$