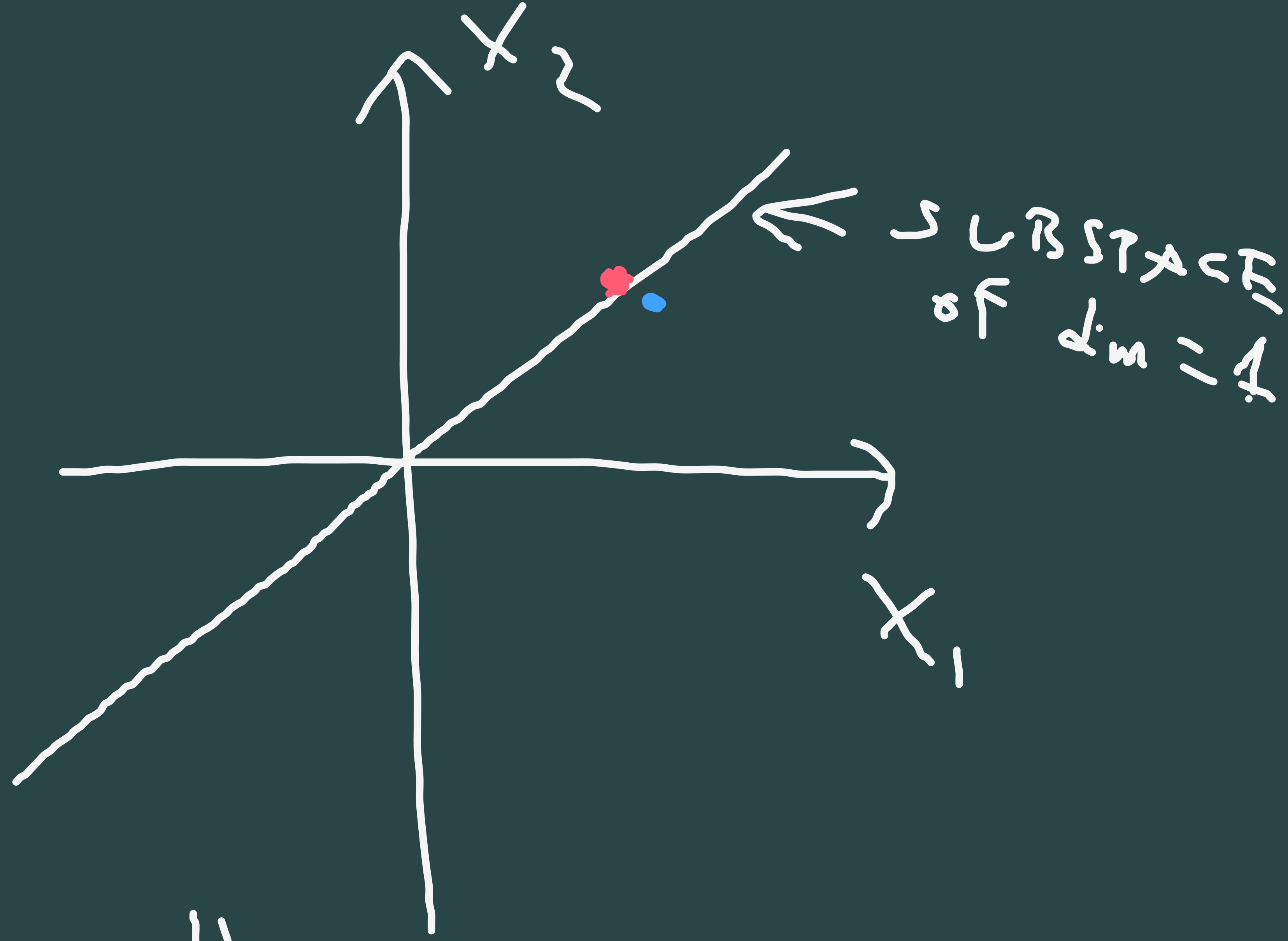
$$A\underline{x} = \underline{b}$$

$\square\, 1 = 1$



Find $\underline{x}$ that minimizes $\|A\underline{x}-\underline{b}\|_2^2$

(LSP)

$$A^T A \underline{x} = A^T \underline{b} \quad (NE)$$

$$\square \, 1 = 1$$

LEMMA

If $A \in \mathbb{R}^{m \times n}$, then $N(A^T A) = N(A)$.

Proof

If $\underline{x} \in N(A) \Rightarrow A\underline{x} = \underline{0} \Rightarrow A^T A \underline{x} = \underline{0} \Rightarrow \underline{x} \in N(A^T A)$

If $\underline{y} \in N(A^T A) \Rightarrow A^T A \underline{y} = \underline{0} \Rightarrow \underline{y}^T A^T A \underline{y} = 0 \Rightarrow \|A\underline{y}\|_2^2 = 0 \Rightarrow A\underline{y} = \underline{0} \Rightarrow \underline{y} \in N(A)$

THM

Assume that (NE) have at least one solution.

Then, $\underline{x}$ is a minimum of (LSP) if and only if it is solves (NE).

Proof

Let $\underline{x} \in \mathbb{R}^n$ solves (NE). Then, for every $\underline{z} \in \mathbb{R}^n$,

$$\underline{z} = \underline{x} + (\underline{z} - \underline{x}). \text{ Hence,}$$

$$\|A\underline{z} - \underline{b}\|_2^2 = \|(A\underline{x} - \underline{b}) + A(\underline{z} - \underline{x})\|_2^2 =$$

$$= \left[(A\underline{x} - \underline{b}) + A(\underline{z} - \underline{x})\right]^T \left[(A\underline{x} - \underline{b}) + A(\underline{z} - \underline{x})\right.$$

$$(\underline{a}+\underline{c})^T(\underline{a}+\underline{c}) = \underline{a}^T\underline{a} + \underline{c}^T\underline{a} + \underline{a}^T\underline{c} + \underline{c}^T\underline{c}$$

$$= \|\underline{a}\|_2^2 + \|\underline{c}\|_2^2 + 2\underline{c}^T\underline{a} \qquad (\text{FACT for all } \underline{a},\underline{b})$$

$$\|A\underline{z}-\underline{b}\|_2^2 = \|A\underline{x}-\underline{b}\|_2^2 + 2(\underline{z}-\underline{x})^T \underbrace{A^T(A\underline{x}-\underline{b})}_{A^TA\underline{x}-A^T\underline{b}=\underline{0}} + \|A(\underline{z}-\underline{x})\|_2^2$$

$$= \|A\underline{x}-\underline{b}\|_2^2 + \|A(\underline{z}-\underline{x})\|_2^2 \geq \|A\underline{x}-\underline{b}\|_2^2$$

$\underline{z}$ is also a minimum of (LSP) iff $\|A(\underline{z}-\underline{x})\|_2 = 0$

iff $A(\underline{z}-\underline{x}) = \underline{0}$ iff $\underline{z}-\underline{x} \in N(A)$

iff $\underline{z}-\underline{x} \in N(A^TA) \Longrightarrow A^TA\underline{z} = A^TA(\underline{x}+\underline{z}-\underline{x}) = A^T\underline{b}+\underline{0}$

$\underline{x}$ minimizes $\|A\underline{x}-\underline{b}\|_2^2$ then $A^TA\underline{x} = A^T\underline{b}$

$$\not\Updownarrow$$

$$A\underline{x} = \underline{b} \;?$$

$$A = \begin{pmatrix}1\\0\end{pmatrix} \qquad \underline{b} = \begin{pmatrix}1\\1\end{pmatrix} \qquad \underline{x} = 1$$

$$A^TA = 1 \qquad A^T\underline{b} = 1 \qquad A^TA\underline{x} = A^T\underline{b}$$

DEF

If $\mathcal{U}, \mathcal{V}$ are subspaces of $\mathbb{R}^n$ s.t.

① $\mathcal{U} \cap \mathcal{V} = \{\underline{0}\}$

② $\mathcal{U} + \mathcal{V} = \{\underline{x} \text{ s.t. } \underline{x} = \underline{u} + \underline{v},\ \underline{u} \in \mathcal{U},\ \underline{v} \in \mathcal{V}\} = \mathbb{R}^n$

We say that $\mathbb{R}^n = \mathcal{U} \oplus \mathcal{V}$ is the DIRECT SUM of $\mathcal{U}$ and $\mathcal{V}$.

LEMMA

If $\mathbb{R}^n = \mathcal{U} \oplus \mathcal{V}$ then, for all $\underline{x} \in \mathbb{R}^n$, there exist

unique $\underline{u} \in \mathcal{U}$, $\underline{v} \in \mathcal{V}$ s.t. $\underline{x} = \underline{u} + \underline{v}$.

Proof

Existence is property (2) in the defn. of direct sum.

Suppose $\underline{x} = \underset{\mathcal{U}}{\underline{u}} + \underset{\mathcal{V}}{\underline{v}} = \underset{\mathcal{U}}{\underline{u}_0} + \underset{\mathcal{V}}{\underline{v}_0} \Rightarrow \underset{\mathcal{U}}{\underline{u} - \underline{u}_0} = \underset{\mathcal{V}}{\underline{v}_0 - \underline{v}} \in \mathcal{U} \cap \mathcal{V} \overset{(1)}{=} \{\underline{0}\}$

$\Downarrow$

$\underline{u} = \underline{u}_0$

$\underline{v} = \underline{v}_0$

$$P_{\mathcal{U}} : \underline{x} = \underline{u} + \underline{v} \longrightarrow P_{\mathcal{U}}\underline{x} = \underline{u}$$

$$P_{\mathcal{V}} : \underline{x} = \underline{u} + \underline{v} \longrightarrow P_{\mathcal{V}}\underline{x} = \underline{v}$$

$$\mathcal{U} = \text{span}(\underbrace{\underline{u}_1, \dots, \underline{u}_k}_{\text{basis}}) \qquad \mathcal{V} = \text{span}(\underbrace{\underline{v}_1, \dots, \underline{v}_{n-k}}_{\text{basis}})$$

$U = [\underline{u}_1, \dots, \underline{u}_k]$ Any $\underline{u} \in \mathcal{U}$ satisfies $U\underline{\alpha} = \underline{u}$ for some $\underline{\alpha}$

$V = [\underline{v}_1, \dots, \underline{v}_{n-k}]$ Any $\underline{v} \in \mathcal{V}$ satisfies $\underline{v} = V\underline{\beta}$ for some $\underline{\beta}$

By (2), $\underline{x} = \underline{u} + \underline{v} = U\underline{\alpha} + V\underline{\beta} = [U \; V] \begin{bmatrix} \underline{\alpha} \\ \underline{\beta} \end{bmatrix}$

So $\mathbb{R}^n = \operatorname{span}(\underline{u}_1, \dots, \underline{u}_k, \underline{v}_1, \dots, \underline{v}_{n-k})$

$$[U\ V]\begin{bmatrix}\underline{\alpha}\\ \beta\end{bmatrix} = \underline{0} = U\underline{\alpha} + V\beta$$

$$\begin{array}{ccccc} U\underline{\alpha} & = & V(-\beta) & \in & \mathcal{U}\cap\mathcal{V} = \{\underline{0}\} \text{ by } (1) \\ \cap & & \cap & & \Downarrow \\ \mathcal{U} & & \mathcal{V} & & U\underline{\alpha} = \underline{0} = V\beta \end{array}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\underline{\alpha} = \underline{0} \qquad \beta = \underline{0}$$

$\underline{x} \in \mathbb{R}^n$ can be written as $\underline{x} = \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} \underline{\alpha} \\ \hline \underline{\beta} \end{bmatrix}$

$= U\underline{\alpha} + V\underline{\beta}$

$P_u \underline{x} = U\underline{\alpha}$

$\begin{bmatrix} \alpha \\ \hline \beta \end{bmatrix} = \begin{bmatrix} U & V \end{bmatrix}^{-1} \underline{x}$

$\underline{\alpha} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \hline \beta \end{bmatrix}$

$U \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \hline \beta \end{bmatrix} = U \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix}^{-1} \underline{x} \Rightarrow P_u = U \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix}^{-1}$

$\parallel$

$U\underline{\alpha}$

$= \begin{bmatrix} U & 0 \end{bmatrix} \begin{bmatrix} U & V \end{bmatrix}^{-1}$

$$P_{\mathcal{V}}\underline{x} = V\beta \qquad P_{\mathcal{V}} = V\begin{bmatrix} 0 & I \end{bmatrix}\begin{bmatrix} U & V \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & V \end{bmatrix}\begin{bmatrix} U & V \end{bmatrix}^{-1}$$

If $\underline{x} \in U$, then $P_U \underline{x} = \underline{x}$

or equivalently, for $\underline{\text{any}}$ $\underline{x} \in \mathbb{R}^n$

$$P_U^2 \underline{x} = P_U \cdot \underbrace{P_U \underline{x}}_{\in U} = P_U \underline{x}$$

DEF

If $P \in \mathbb{R}^{n \times n}$ is s.t. $P^2 = P$ then it is called a PROJECTION MATRIX.

$U = R(P)$, $V = R(I-P)$.

If P is a projection matrix, $\mathbb{R}^n = U \oplus V$

Proof of the claim

(2) $$\underline{x} = \underbrace{P\underline{x}}_{\in R(P)} + \underbrace{(I-P)\underline{x}}_{\in R(I-P)}$$

To prove (1), suppose $\underline{x} \in R(P) \cap R(I-P)$. then

$$\underline{x} = P\underline{x} \quad \underline{x} = (I-P)\underline{z}$$

$$\underline{x} = P\underline{x} = P(I-P)\underline{z} = (P-P^2)\underline{z} = 0\underline{z} = \underline{0}$$

$$\Rightarrow R(P) \cap R(I-P) = \{\underline{0}\}$$

$$\underline{x} = P\underline{v}$$

$$P\underline{x} = P^2\underline{v} = P\underline{v} = \underline{x}$$

Def

Let $V \subseteq \mathbb{R}^n$ be a subspace. The ORTHOGONAL COMPLEMENT of V wrt $\langle \cdot, \cdot \rangle$

$$V^\perp = \{ \underline{w} \in \mathbb{R}^n \text{ s.t. } \langle \underline{v}, \underline{w} \rangle = 0 \text{ for all } \underline{v} \in V \}$$

Exercise:

$V^{\perp}$ is a subspace.