Advanced Microeconomics 1
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Final Examination Suggested solutions

1. Short answers below.

   (a) A preference relation $\succeq$ on domain I is complete if for all $i, j \in I$, $i \succeq j$ or $j \succeq i$ or both. It is transitive if for all $i, j, k \in I$, $i \succeq j$ and $j \succeq k$ imply $i \succeq k$. The given preference is complete since for all i, j either $x_i > x_j + 1$, $x_j > x_i + 1$ or $|x_i - x_j| \leq 1$. In the first case $i \succeq j$, in the second $j \succeq i$. In the third, $i \succeq j$ if and only if $y_i \geq y_j$. Since $y_i \geq y_j$ or $y_j \geq y_i$ or both, we see that in the third case either $i \succeq j$ or $j \succeq i$. The preference is not transitive on the domain of all possible candidates. For example, take $(x_1, y_1) = (1, 3)$ $(x_2, y_2) = (2, 2)$ $(x_3, y_3) = (3, 1)$. Then $1 \succeq 2$, $2 \succeq 3$, but it is not the case that $1 \succeq 3$. (If you have a fixed set I of candidates, then transitivity may hold e.g. if $|x_i - x_j| > 1$ for all $i \neq j$).

   (b) A competitive firm maximizes its profit $\boldsymbol{p} \cdot \boldsymbol{y}$ at given prices $\boldsymbol{p}$ by choosing a production vector $\boldsymbol{y}$ in the production set Y. Let $\boldsymbol{y}$ be the optimal production at $\boldsymbol{p}$ and $\boldsymbol{y}'$ optimal production at $\boldsymbol{p}'$. Then since $\boldsymbol{y}, \boldsymbol{y}' \in Y$, we have:

   $$\boldsymbol{p} \cdot \boldsymbol{y} \geq \boldsymbol{p} \cdot \boldsymbol{y}' \text{ and } \boldsymbol{p}' \cdot \boldsymbol{y}' \geq \boldsymbol{p}' \cdot \boldsymbol{y}.$$

   Summing the inequalities, we get:

   $$(\boldsymbol{p} - \boldsymbol{p}') \cdot (\boldsymbol{y} - \boldsymbol{y}') \geq 0.$$

   Let k be an output for the firm and let $\boldsymbol{p}' = \boldsymbol{p} + \Delta \boldsymbol{e}_k$ with $\Delta > 0$. Then we have:

   $$-\Delta(y_k - y_k') \geq 0,$$

   implying that $y_k' \geq y_k$.

   (c) Demand function $\boldsymbol{x}(\boldsymbol{p}, w)$ satisfies Walras' law if $\boldsymbol{p} \cdot \boldsymbol{x}(\boldsymbol{p}, w) = w$ for all $\boldsymbol{p}, w$. A set of observed demands at $\{\boldsymbol{x}_i\}_{i=1}^K$ at prices $\{\boldsymbol{p}_i\}_{i=1}^K$ satisfies WARP if $\boldsymbol{p}_i \cdot \boldsymbol{x}_j \leq \boldsymbol{p}_i \cdot \boldsymbol{x}_i$ implies $\boldsymbol{p}_j \cdot \boldsymbol{x}_i > \boldsymbol{p}_j \cdot \boldsymbol{x}_j$ for all $\boldsymbol{x}_i \neq \boldsymbol{x}_j$. Here $\boldsymbol{p} \cdot \boldsymbol{x} = \boldsymbol{p}' \cdot \boldsymbol{x}' = 2 = w$ so demands satisfy Walras' law. $\boldsymbol{p} \cdot \boldsymbol{x}' = 4/3 < 2$, and $\boldsymbol{p}' \cdot \boldsymbol{x} = 3 > 2$. So $\boldsymbol{x}$ is revealed preferred to $\boldsymbol{x}'$, and $\boldsymbol{x}'$ is not revealed preferred to $\boldsymbol{x}$ and the demands satisfy WARP.

2. A consumer has a quasilinear utility function $u(x, y) = x + v(y)$ for some strictly concave and strictly increasing function v.

(a) Since u is strictly increasing, budget constraint holds as equality at any optimum, i.e. $x = \frac{w - yp_y}{p_x}$. Consider then the problem:

$$\max_{y \in [0, \frac{w}{p_y}]} \frac{w - yp_y}{p_x} + v(y).$$

Since the objective function is strictly concave in y, first-order conditions are sufficient for maximum $y^* = y(p_x, p_y, w)$.

$$\text{If } 0 < y^* < \frac{w}{p_y}, \text{ then } v'(y^*) = \frac{p_y}{p_x}.$$

$$\text{If } y^* = 0, \text{ then } v'(0) \leq \frac{p_y}{p_x}.$$

$$\text{If } y^* = \frac{w}{p_y}, \text{ then } v'(\frac{w}{p_y}) \geq \frac{p_y}{p_x}.$$

(b) For interior solutions,

$$y(1, p_y, w) = (v')^{-1}(p_y).$$

Hence $\frac{\partial y(1,p_y,w)}{\partial w} = 0$. By Walras' law, $\frac{\partial x(1,p_y,w)}{\partial w} = 1$. Implicit function theorem on the interior first-order condition $v'(y^*) = p_y$ gives $\frac{\partial y(1,p_y,w)}{\partial p_y} = \frac{1}{v''(y^*)}$. This is negative by the strict concavity of v. By Walras' law, $x = w - p_y y(1, p_y, w)$. Hence $\frac{\partial x(1,p_y,w)}{\partial p_y} \geq 0$ if and only the demand for y is elastic in its own price.

(c) The interior first order condition $\frac{1}{y^*} = p_y$ gives $y^* = \frac{2}{3}$, $x^* = 1$ at $p_y = \frac{3}{2}$. This demand is feasible at both uncertain prices and the expected utility with this constant demand is the same as with certain prices:

$$\frac{1}{2}(\frac{4}{3} + \ln\left(\frac{2}{3}\right)) + \frac{1}{2}(\frac{2}{3} + \ln\left(\frac{2}{3}\right)) = 1 + \ln\left(\frac{2}{3}\right).$$

By demanding a bit more of y at $p_y = 1$ and a bit less at $p_y = 2$, the consumer can strictly improve her utility. (Alternatively, you could compute the optimal demands for each price and compare the expected utility to certain utility and get the same result).

3. Answers below.

(a) Final wealth $\tilde{w}$ is $\alpha_1 w_0 R^H + \alpha_2 w_0 R$ with probability p and it is $\alpha_2 w_0 R$ with probability $(1 - p)$.

(b) Expected utility:

$$v(\alpha_1, \alpha_2) = pu(\alpha_1 w_0 R^H + \alpha_2 w_0 R) + (1 - p)u(\alpha_2 w_0 R).$$

Observe that since u is strictly increasing, the sum of the alphas is 1 at optimum. Hence the first order condition for an interior solution is:

$$w_0(R^H - R)pu'(\alpha_1 w_0 R^H + (1-\alpha_1)w_0 R) - w_0 R(1-p)u'((1-\alpha_1)w_0 R) = 0.$$

We see a necessary and sufficient condition for an interior solution is that $pR^H > R$. First order conditions are sufficient since the expected utility $v(\alpha_1)$ is strictly concave in α_1.

(c) Expected utility from investing α in asset 1 and $(1-\alpha)$ in asset 3 is:

$$v(\alpha) = p\ln\big(\alpha w_0 R^H\big) + (1-p)\ln\big((1-\alpha)w_0 R^L\big).$$

First order condition:

$$\frac{pw_0R^H}{\alpha w_0 R^H} = \frac{(1-p)w_0R^L}{(1-\alpha)w_0R^L}.$$

We get:

$$\frac{\alpha}{1-\alpha} = \frac{p}{1-p}.$$

So at optimum: $\alpha = p$.

(d) (Extra credit) When $R^L \neq R^H$, we get from the FOC above the same conclusion for the optimal share if only 1 and 3 are used. Hence we need only compare the marginal utility of buying a little of the certain asset at this optimal portfolio:

$$\frac{pw_0R}{\alpha w_0R^H} + \frac{(1-p)w_0R}{(1-\alpha)w_0R^L}$$

to the marginal utility of buying more of asset 1 at this optimal portfolio. But the latter marginal utility is 1 from part (c). Hence we have a simple comparison of $\frac{R}{R^H} + \frac{R}{R^L}$ to 1. It is optimal to invest exclusively in 1 and 3 if:

$$R \leq \frac{R^H R^L}{R^H + R^L}.$$

Notice that this does not depend on p.