TIØ 1: Financial Engineering in Energy Markets

Afzal Siddiqui
Department of Statistical Science
University College London
London WC1E 6BT, UK
afzal@stats.ucl.ac.uk

UCL

COURSE OUTLINE

- ★ Introduction (Chs 1–2)
- ★ Mathematical Background (Chs 3–4)
- ★ Investment and Operational Timing (Chs 5–6)
- ★ Entry, Exit, Lay-Up, and Scrapping (Ch 7)
- ★ Recent Theoretical Work I: Capacity Sizing
- ★ Recent Theoretical Work II: Risk Aversion and Multiple Risk Factors
- ★ Applications to the Energy Sector I: Capacity Sizing, Timing, and Operational Flexibility
- ★ Applications to the Energy Sector II: Modularity and Technology Choice

LECTURE OUTLINE

- ★ Basic model
- ★ Solutions via dynamic programming and contingent claims
- ★ Characteristics of optimal investment
- ★ Alternative stochastic processes
- ★ Operating costs and temporary suspension
- ★ Projects with variable output
- ★ Depreciation
- ★ Price and cost uncertainty

BASIC MODEL: Optimal Timing

- ★ Suppose project value, V, evolves according to a GBM, i.e., $dV = \alpha V dt + \sigma V dz$, which may be obtained at a sunk cost of I

- ★ When is the optimal time to invest?
  - ▶ A perpetual option, i.e., calendar time is not important
  - ▶ Ignore temporary suspension or other embedded options
  - ▶ Use both dynamic programming and contingent claims methods

- ★ Problem formulation: $\max_T \mathcal{E}_{V_0}\left[(V_T - I)e^{-\rho T}\right]$
  - ▶ Assume $\delta \equiv \rho - \alpha > 0$, otherwise it is always better to wait indefinitely

BASIC MODEL: Deterministic Case

- Suppose that $\sigma = 0$, i.e., $V(t) = V_0 e^{\alpha t}$ for $V_0 \equiv V(0)$
  - $F(V) \equiv \max_T e^{-\rho T}(Ve^{\alpha T} - I)$
  - If $\alpha \leq 0$, then $F(V) = \max[V - I, 0]$
  - Otherwise, for $0 < \alpha < \rho$, waiting may be better because either (i) $V < I$ or (ii) $V \geq I$, but discounting of future sunk cost is greater than that in the future project value
  - Thus, the FONC is $\frac{dF(V)}{dT} = 0 \Rightarrow (\rho - \alpha)Ve^{-(\rho-\alpha)T} = \rho I e^{-\rho T} \Rightarrow$ $T^* = \max\left\{\frac{1}{\alpha}\ln\left\{\frac{\rho I}{(\rho-\alpha)V}\right\}, 0\right\}$
  - Reason for delaying is that the MC is depreciating over time by more than the MB
- Substitute T^* to determine $V^* = \frac{\rho I}{(\rho-\alpha)} > I$
- And, $F(V) = \left(\frac{\alpha I}{\rho-\alpha}\right)\left[\frac{(\rho-\alpha)V}{\rho I}\right]^{\frac{\rho}{\alpha}}$ if $V \leq V^*$ ($F(V) = V - I$ otherwise)
- Figure 5.1 indicates that greater α increases V^*

BASIC MODEL: Figure 5.1

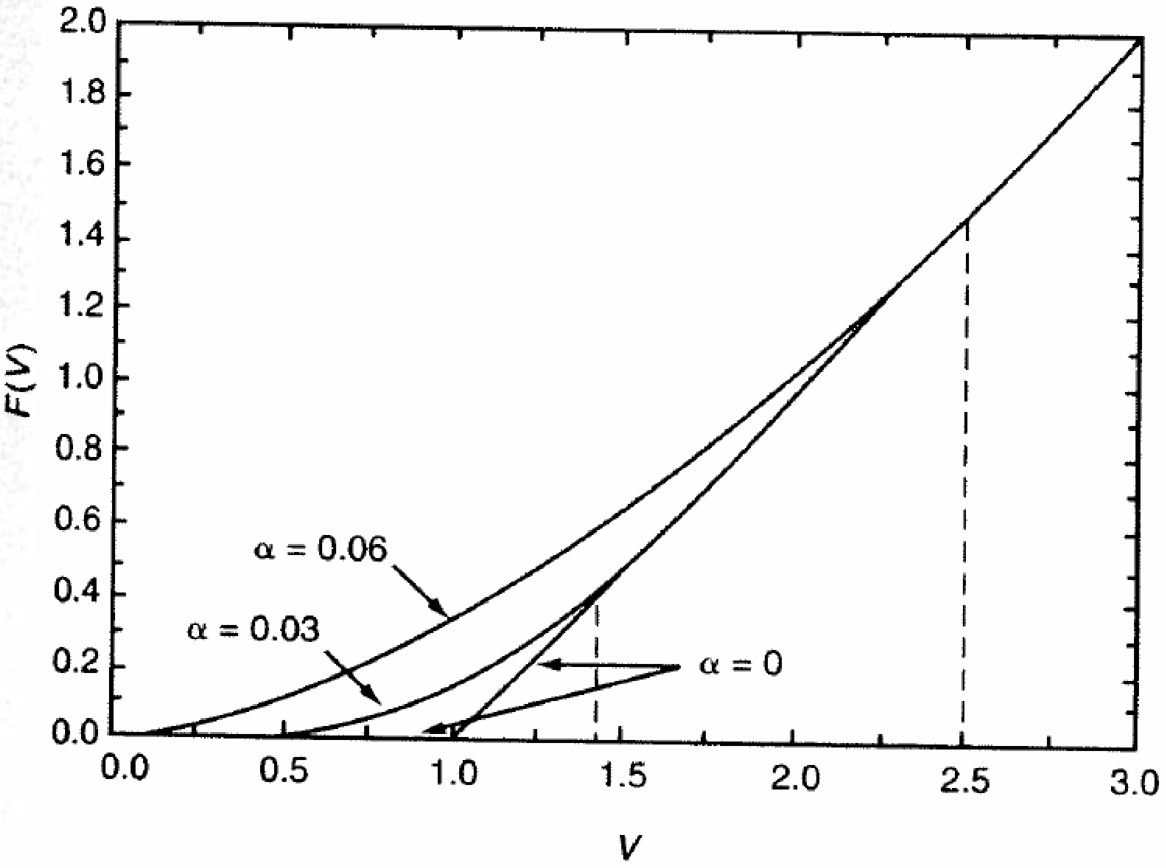


Figure 5.1. *Value of Investment Opportunity, $F(V)$, for $\sigma = 0$, $\rho = 0.1$*

DYNAMIC PROGRAMMING SOLUTION

- ★ Bellman equation for continuation is $\rho F dt = \mathcal{E}[dF]$
- ★ Expand the RHS via Itô's lemma: $dF = F'(V)dV + \frac{1}{2}F''(V)(dV)^2 \Rightarrow \mathcal{E}[dF] = F'(V)\alpha V dt + \frac{1}{2}F''(V)\sigma^2 V^2 dt$
- ★ Substitution into the Bellman equation yields the ODE $\frac{1}{2}F''(V)\sigma^2 V^2 + F'(V)\alpha V - \rho F(V) = 0$
  - ▶ Equivalently, $\frac{1}{2}F''(V)\sigma^2 V^2 + F'(V)(\rho - \delta)V - \rho F(V) = 0$
  - ▶ Three boundary conditions: (i) $F(0) = 0$, (ii) $F(V^*) = V^* - I$, and (iii) $F'(V^*) = 1$
  - ▶ General solution to the ODE is $F(V) = A_1 V^{\beta_1} + A_2 V^{\beta_2}$
  - ▶ Taking derivatives, we have $F'(V) = A_1\beta_1 V^{\beta_1 - 1} + A_2\beta_2 V^{\beta_2 - 1}$ and $F''(V) = A_1\beta_1(\beta_1 - 1)V^{\beta_1 - 2} + A_2\beta_2(\beta_2 - 1)V^{\beta_2 - 1}$
  - ▶ Substitution into the ODE yields $A_1 V^{\beta_1}[\frac{1}{2}\sigma^2\beta_1(\beta_1 - 1) + \beta_1(\rho - \delta) - \rho] + A_2 V^{\beta_2}[\frac{1}{2}\sigma^2\beta_2(\beta_2 - 1) + \beta_2(\rho - \delta) - \rho] = 0$
  - ▶ Thus, $\beta_1 = \frac{1}{2} - \frac{(\rho-\delta)}{\sigma^2} + \sqrt{\left[\frac{\rho-\delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}$ and $\beta_2 = \frac{1}{2} - \frac{(\rho-\delta)}{\sigma^2} - \sqrt{\left[\frac{\rho-\delta}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}}$

DYNAMIC PROGRAMMING SOLUTION

- The characteristic quadratic, $\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1)+(\rho-\delta)\beta-\rho$, has two roots such that $\beta_1 > 1$ and $\beta_2 < 0$
  - $\mathcal{Q}(\beta)$ has a positive coefficient for β^2, i.e., it is an upward-pointing parabola
  - Note that $\mathcal{Q}(1) = -\delta < 0$, which means that $\beta_1 > 1$
  - $\mathcal{Q}(0) = -\rho$, which means that $\beta_2 < 0$ (Figure 5.2)
- Consequently, the first boundary condition implies that $A_2 = 0$, i.e., $F(V) = A_1 V^{\beta_1}$
  - Using the VM and SP conditions, we obtain $V^* = \frac{\beta_1}{\beta_1-1}I$ and $A_1 = \frac{(V^*-I)}{(V^*)^{\beta_1}} = \frac{(\beta_1-1)^{\beta_1-1}}{[(\beta_1)^{\beta_1} I^{\beta_1-1}]}$
  - Since $\beta_1 > 1$, we also have $V^* > I$

DYNAMIC PROGRAMMING SOLUTION: Figure 5.2

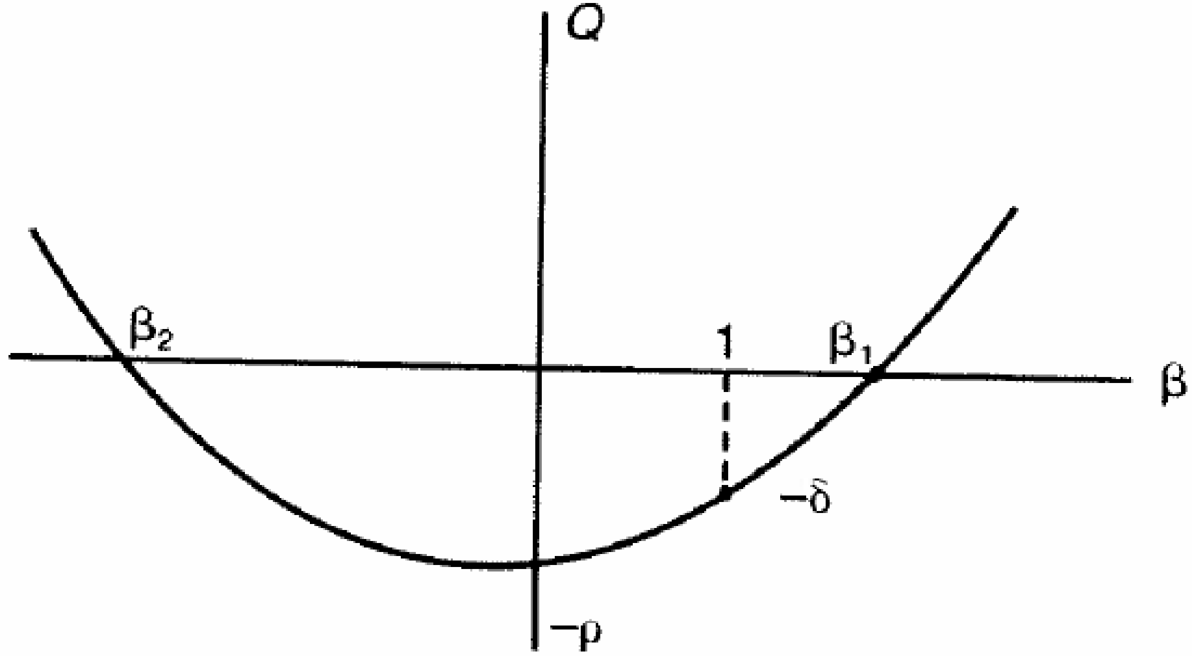


Figure 5.2. *The Fundamental Quadratic*

DYNAMIC PROGRAMMING SOLUTION: Comparative Statics

- $\frac{\partial \beta_1}{\partial \sigma} < 0$
  - Differentiate $\mathcal{Q}(\beta)$ totally and evaluate it at β_1
  - $\frac{\partial \mathcal{Q}}{\partial \beta}\frac{\partial \beta_1}{\partial \sigma} + \frac{\partial \mathcal{Q}}{\partial \sigma} = 0 \Rightarrow \frac{\partial \beta_1}{\partial \sigma} = -\frac{\partial \mathcal{Q}/\partial \sigma}{\partial \mathcal{Q}/\partial \beta}$
  - Know that $\frac{\partial \mathcal{Q}}{\partial \beta} > 0$ at β_1 via Figure 5.2 and $\frac{\partial \mathcal{Q}}{\partial \sigma} = \sigma\beta(\beta-1) > 0$ at $\beta_1 > 1$
  - Thus, $\frac{\partial \beta_1}{\partial \sigma} < 0$ and $\frac{\beta_1}{\beta_1 - 1}$ increases with σ
- Similarly, $\frac{\partial \beta_1}{\partial \delta} = -\frac{\partial \mathcal{Q}/\partial \delta}{\partial \mathcal{Q}/\partial \beta} > 0$
  - For $\beta_1 > 1$, $\frac{\partial \mathcal{Q}}{\partial \delta} = -\beta < -1$
  - Thus, $\frac{\partial \beta_1}{\partial \delta} > 0$ and $\frac{\beta_1}{\beta_1 - 1}$ decreases with δ
- Finally, $\frac{\partial \beta_1}{\partial \rho} = -\frac{\partial \mathcal{Q}/\partial \rho}{\partial \mathcal{Q}/\partial \beta} < 0$
  - For $\beta_1 > 1$, $\frac{\partial \mathcal{Q}}{\partial \rho} = \beta - 1 > 0$
  - Thus, $\frac{\partial \beta_1}{\partial \rho} < 0$ and $\frac{\beta_1}{\beta_1 - 1}$ increases with ρ
- As $\sigma \to \infty$, $\beta_1 \to 1$ and $V^* \to \infty$, whereas as $\sigma \to 0$, $\beta_1 \to \frac{\rho}{\rho - \delta}$ and $V^* \to \frac{\rho}{\delta} I$ for $\alpha > 0$

DYNAMIC PROGRAMMING SOLUTION: Comparison to Neoclassical Theory

- Marshallian analysis is to compare $V_0 \equiv \mathcal{E}_{\pi_0} \int_0^\infty \pi_s e^{-\rho s} ds = \int_0^\infty \mathcal{E}_{\pi_0}[\pi_s] e^{-\rho s} ds = \frac{\pi_0}{\rho - \alpha}$ with I
  - Invest if $V_0 \geq I$ or $\pi_0 \geq (\rho - \alpha) I$
  - Real options approach says to invest when $\pi_0 \geq \pi^* \equiv \frac{\beta_1}{\beta_1 - 1}(\rho - \alpha) I > (\rho - \alpha) I$

- Tobin's q is the ratio of the value of the existing capital goods to the their current reproduction cost
  - Rule is to invest when $q \geq 1$
  - If we interpret q as being $\frac{V}{I}$, then the real options threshold is $q^* = \frac{\beta_1}{\beta_1 - 1} > 1$
  - Hence, the real options definition of q adds option value to the PV of assets in place

CONTINGENT CLAIMS SOLUTION: Background

★ Instead of using an arbitrary discount rate, ρ, we now try to ground it more firmly using market principles

- Assume that x is the price of an asset that is perfectly correlated with V, i.e., $\rho_{xm} = \rho_{VM}$
- If x pays no dividends, then $dx = \mu x dt + \sigma x dz$
- From CAPM, $\mu = r + \phi \rho_{xm} \sigma > \alpha$, where α is the expected percentage rate of change of V
- Let $\delta = \mu - \alpha$ be the dividend rate, and if it were equal to zero, then it would imply that the option would always be held to maturity
- In other words, there would be no opportunity cost to delaying exercise of the option since the entire return comes from the price movement, i.e., one would never invest
- Thus, we assume $\delta > 0$, and if $\delta \to \infty$, then invest either now or never, i.e., opportunity cost of waiting is high and options value goes to zero

CONTINGENT CLAIMS SOLUTION

- Find $F(V)$ by constructing a risk-free portfolio, Φ, which consists of one unit of $F(V)$ and $n = F'(V)$ units short of the underlying project (or correlated asset)
  - Recall from the previous lecture that $n = \frac{bF_x}{BX}$ in order for the synthetic portfolio to be risk free
  - $\Phi = F - F'(V)V$, which means that n must change over time even if it is kept constant for the next dt time units
  - Short position requires dividend payment of $\delta V F'(V)$
  - Thus, the total portfolio return is $d\Phi - \delta F'(V)V dt = dF - F'(V)dV - \delta F'(V)V dt$
  - From Itô's lemma, we have $dF = F'(V)dV + \frac{1}{2}F''(V)(dV)^2$
  - Substitution yields the total portfolio return is $\frac{1}{2}F''(V)(dV)^2 - \delta F'(V)V dt = \frac{1}{2}F''(V)V^2\sigma^2 dt - \delta F'(V)V dt$
  - The no-arbitrage condition implies $\frac{1}{2}F''(V)V^2\sigma^2 dt - \delta F'(V)V dt = r[F - F'(V)V]dt \Rightarrow \frac{1}{2}F''(V)V^2\sigma^2 + (r-\delta)F'(V)V - rF = 0$
  - Hence, $F(V) = A_1 V^{\beta_1}$, where $\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}$

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE

★ Use numerical examples to illustrate how investment values and thresholds change using $I = 1$, $r = 0.04$, $\delta = 0.04$, and $\sigma = 0.20$

- This implies that $\beta_1 = 2$, $V^* = 2I = 2$, and $A_1 = \frac{1}{4}$, i.e., real options says to invest when project value is twice as high as the investment cost
- Furthermore, $F(V) = \frac{1}{4}V^2$ for $V \leq 2$ and $F(V) = V - 1$ otherwise (Figure 5.3)
- Note that $F(V)$ and V^* increase with σ: greater uncertainty increases value of waiting and, thus, the opportunity cost of investing (Figure 5.4)
- Greater δ increases the opportunity cost of delaying the investment and, thus, reduces the option value and the investment threshold (Figures 5.5 and 5.6)
- Caveat: σ and δ are related via $\delta = \mu - \alpha = r + \phi\sigma\rho_{xm} - \alpha$, but we treat them as being independent for sake of exposition

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE: Figure 5.3

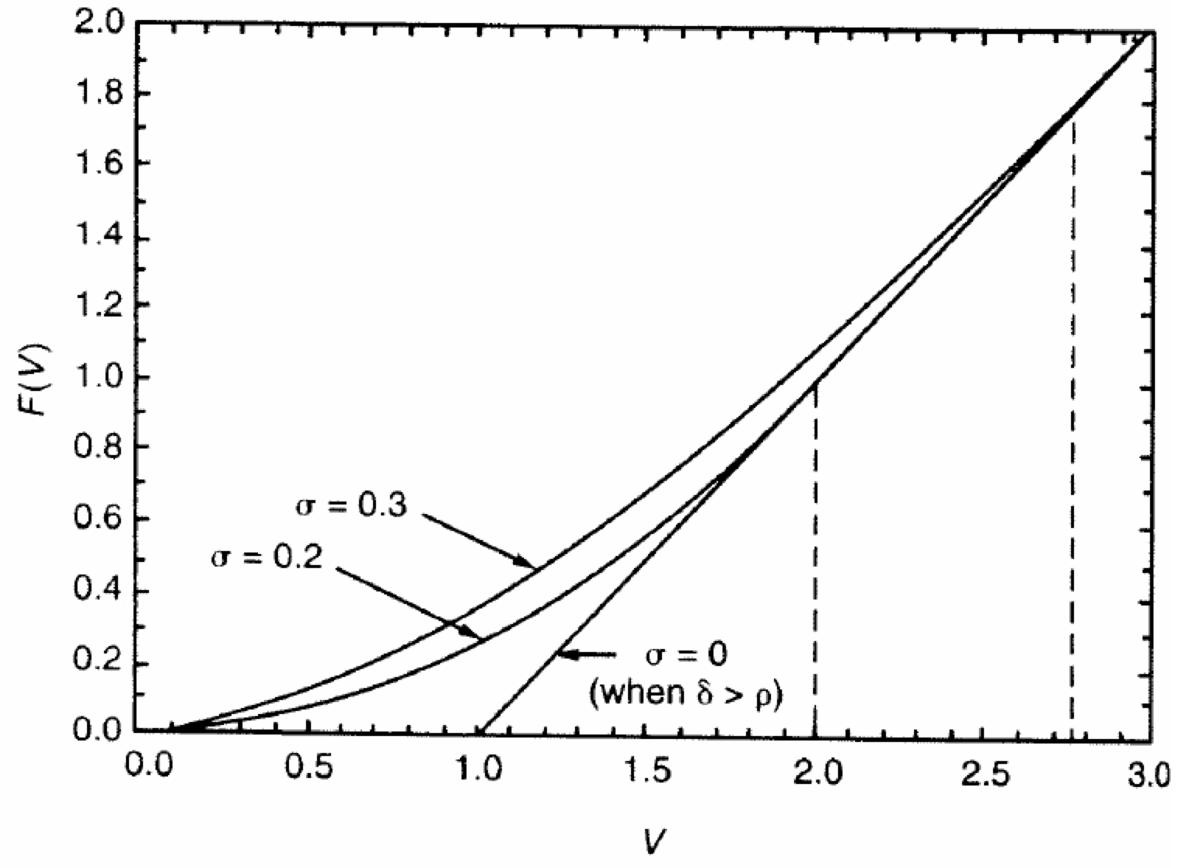


Figure 5.3. *Value of Investment Opportunity, $F(V)$, for $\sigma = 0$, 0.2, and 0.3*

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE: Figure 5.4

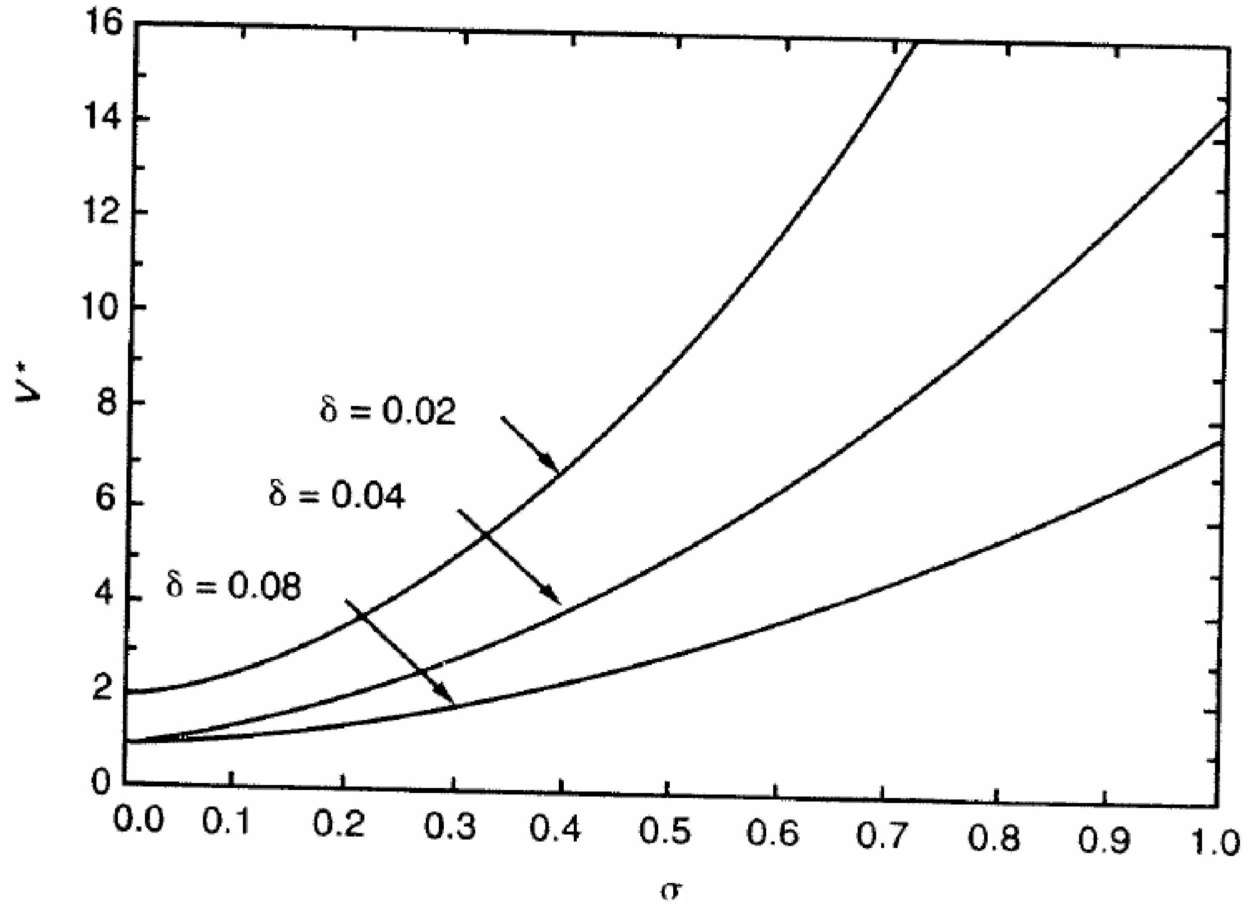


Figure 5.4. *Critical Value V^* as a Function of σ*

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE: Figure 5.5

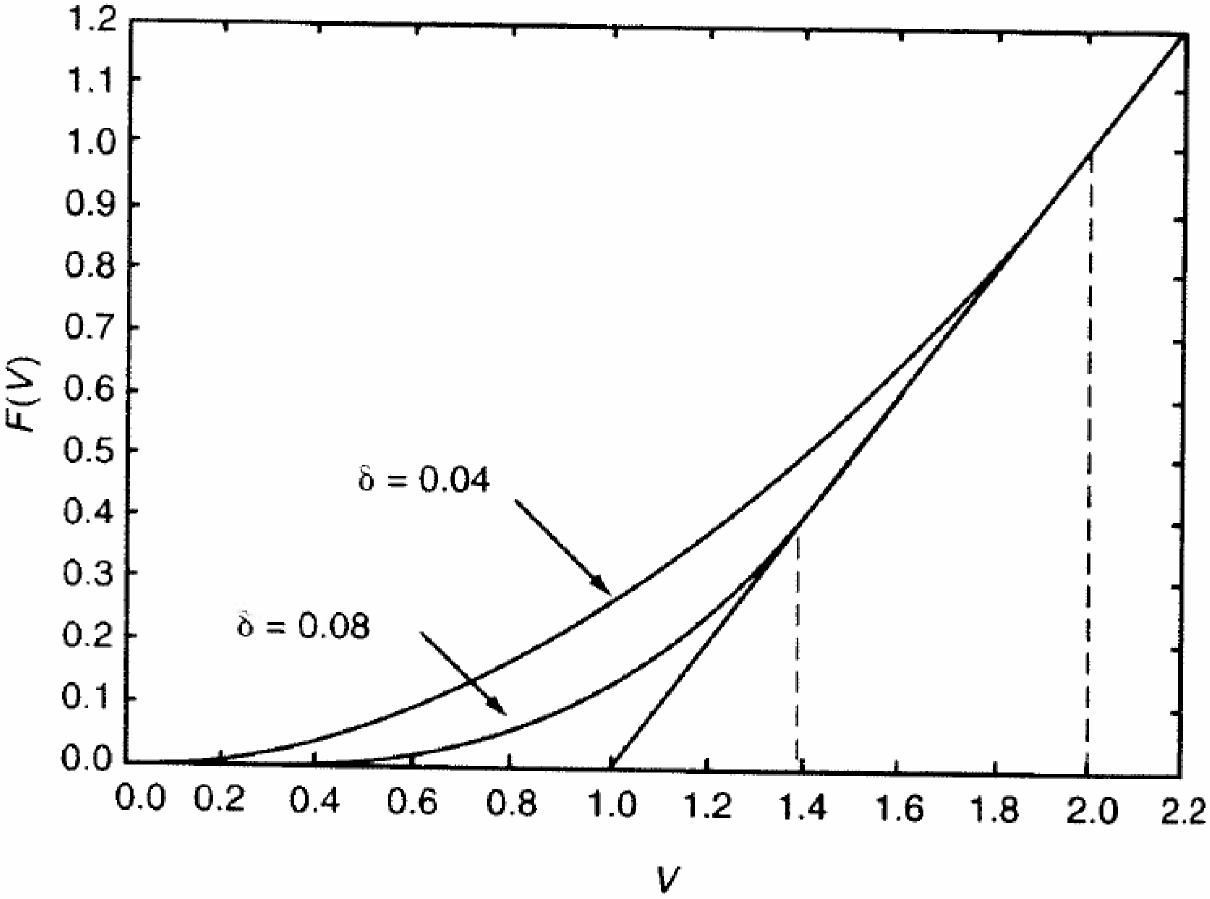


Figure 5.5. Value of Investment Opportunity, $F(V)$, for $\delta = 0.04$ and 0.08

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE: Figure 5.6

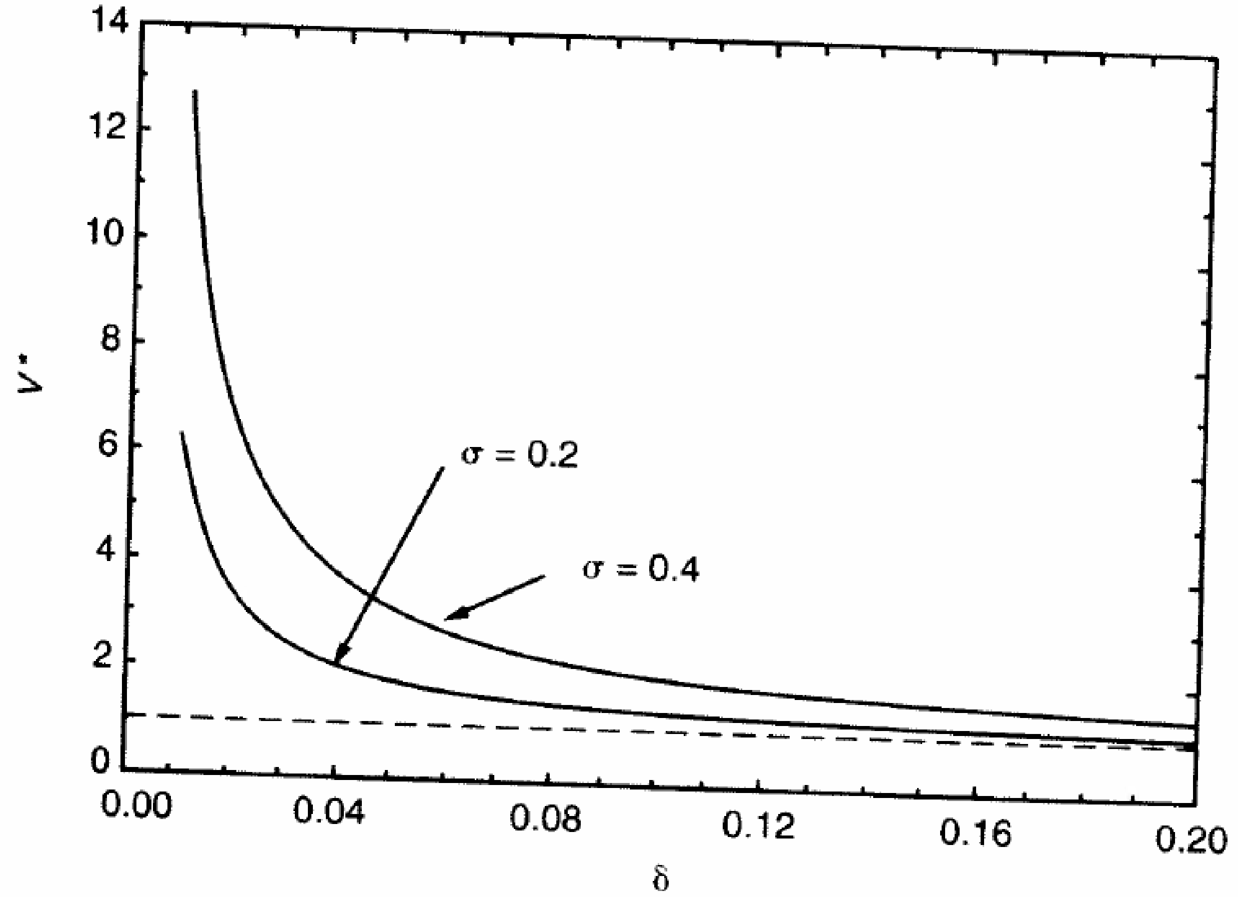


Figure 5.6. Critical Value V^* as a Function of δ

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE

★ Use numerical examples to illustrate how investment values and thresholds change using $I = 1$, $r = 0.04$, $\delta = 0.04$, and $\sigma = 0.20$

- Increasing r increases $F(V)$ and V^* because the PV of expenditure at future time, T, Ie^{-rT}, is reduced while the PV of revenue, $Ve^{-\delta T}$, is unaffected (Figure 5.7)
- Thus, it is worthwhile to wait more even if the value of the option increases
- Cast results in terms of Tobin's $q = \frac{V^*}{I} = \frac{\beta_1}{\beta_1 - 1}$, i.e., use definition without option value
- Plot contours of constant q^* for combinations of $\frac{2r}{\sigma^2}$ and $\frac{2\delta}{\sigma^2}$ (Figure 5.8)
- Find that q^* is large when either δ is small or r is large: intuitively, higher dividend rate reduces value of waiting, while higher interest rate does the opposite
- Finally, note that all estimated parameters, such as α and σ, may be changing over time

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE: Figure 5.7

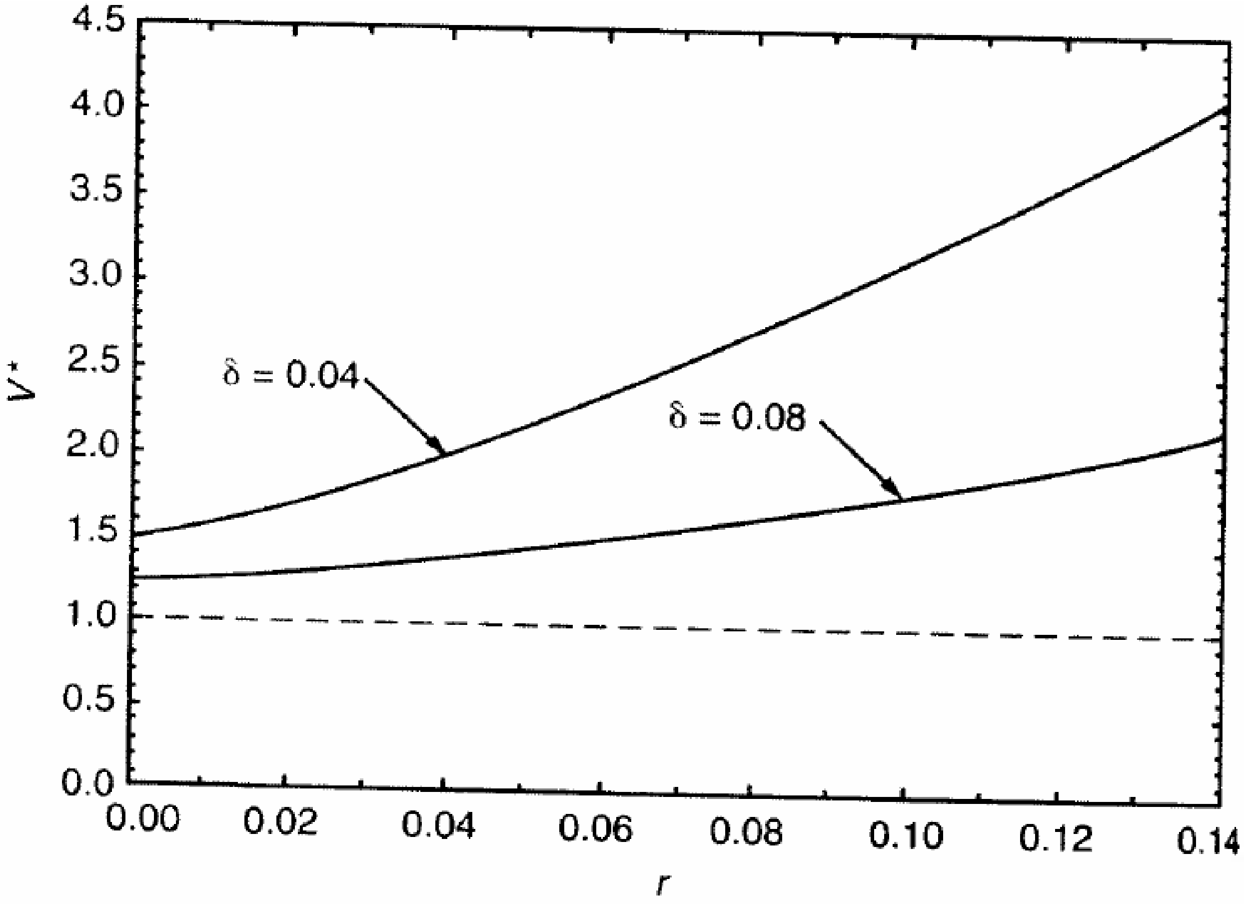


Figure 5.7. Critical Value V^* as a Function of r

CHARACTERISTICS OF THE OPTIMAL INVESTMENT RULE: Figure 5.8

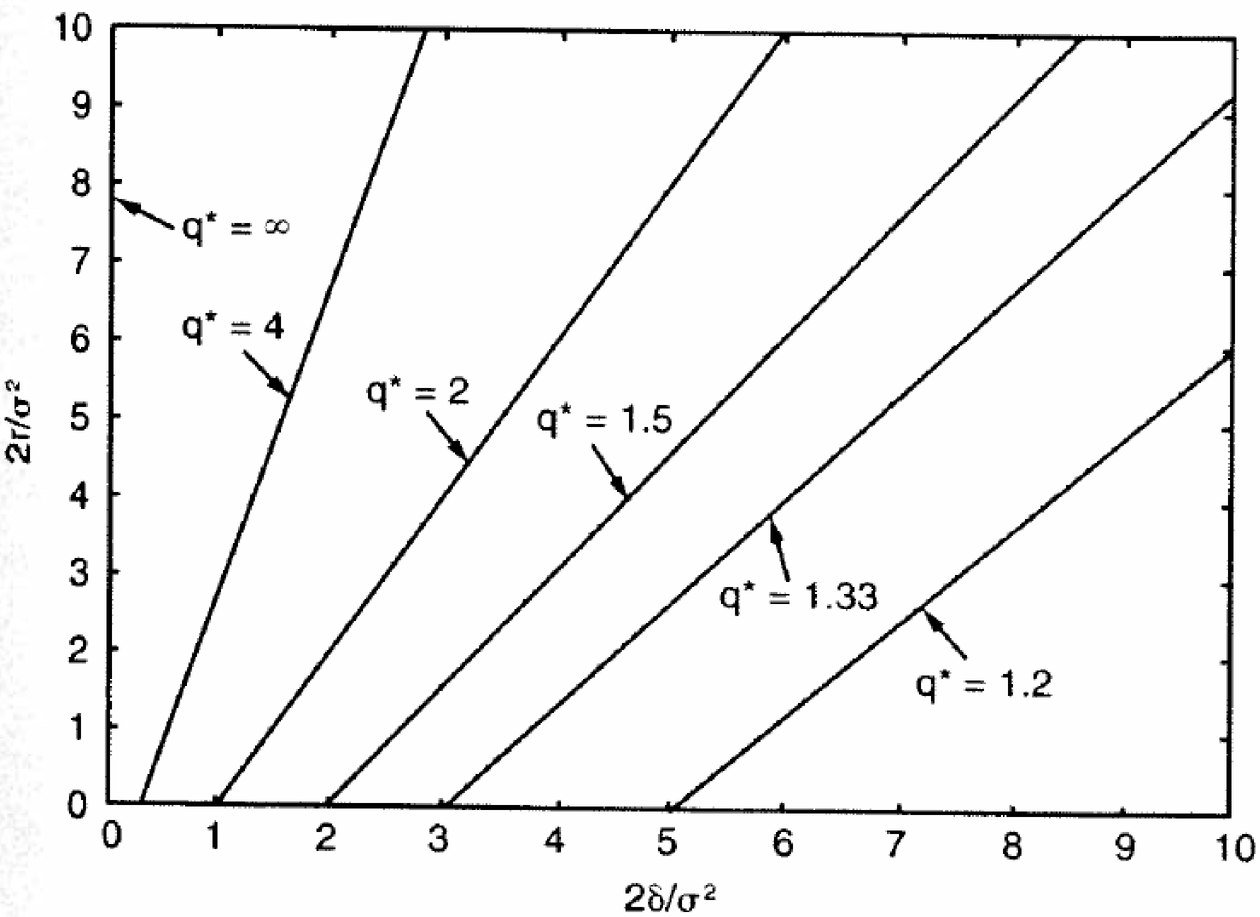


Figure 5.8. *Curves of Constant* $q^* = \beta_1/(\beta_1 - 1)$

ALTERNATIVE STOCHASTIC PROCESSES: GMR Process

- Suppose V follows a GMR process: $dV = \eta(\overline{V} - V)V\,dt + \sigma V\,dz$
  - Expected percentage change of V is $\frac{1}{dt}\mathcal{E}\left[\frac{dV}{V}\right] = \eta(\overline{V} - V)$
  - Thus, expected absolute rate of change is $\frac{1}{dt}\mathcal{E}\left[dV\right] = \eta V\overline{V} - \eta V^2$, which is a parabola that is zero at $V = 0$ and $V = \overline{V}$ with a maximum at $\frac{\overline{V}}{2}$
  - Let μ be the risk-adjusted rate of return for the project and define the dividend rate to be $\delta(V) = \mu - \frac{1}{dt}\mathcal{E}\left[\frac{dV}{V}\right] = \mu - \eta(\overline{V} - V)$
  - End up with same ODE as before using contingent claims, but adjust for $\delta(V)$: $\frac{1}{2}\sigma^2 V^2 F''(V) + [r - \mu + \eta(\overline{V} - V)]VF'(V) - rF = 0$
  - Must satisfy the same three boundary conditions as before
  - Typically, a closed-form solution is difficult to find
  - Express the solution as $F(V) = AV^{\theta}h(V)$ and substitute it back into the ODE

ALTERNATIVE STOCHASTIC PROCESSES: GMR Process

★ Since $F'(V) = \theta A V^{\theta-1} h(V) + A V^{\theta} h'(V)$ and $F''(V) = \theta(\theta-1) A V^{\theta-2} h(V) + 2\theta A V^{\theta-1} h'(V) + A V^{\theta} h''(V)$

- We have $V^{\theta} h(V)\left[\frac{1}{2}\sigma^2\theta(\theta-1) + (r - \mu + \eta\overline{V})\theta - r\right] + V^{\theta+1}\left[\frac{1}{2}\sigma^2 V h''(V) + (\sigma^2\theta + r - \mu + \eta\overline{V} - \eta V)h'(V) - \eta\theta h(V)\right] = 0$
- Both bracketed components must be zero, i.e., $\frac{1}{2}\sigma^2\theta(\theta-1) + (r - \mu + \eta\overline{V})\theta - r = 0 \Rightarrow \theta = \frac{1}{2} + \frac{(\mu^2 - r - \eta\overline{V})}{\sigma^2} + \sqrt{\left[\frac{r-\mu+\eta\overline{V}}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}}$
- And also $\frac{1}{2}\sigma^2 V h''(V) + (\sigma^2\theta + r - \mu + \eta\overline{V} - \eta V)h'(V) - \eta\theta h(V) = 0$
- Use substitution $x = \frac{2\eta V}{\sigma^2}$ to transform it into Kummer's equation, $x g''(x) + (b - x) g'(x) - \theta g(x)$, which has the solution $H(x;\theta,b) = 1 + \frac{\theta}{b}x + \frac{\theta(\theta+1)x^2}{b(b+1)2!} + \frac{\theta(\theta+1)(\theta+2)x^3}{b(b+1)(b+2)3!} + \cdots$
- Hence, $F(V) = A V^{\theta} H\left(\frac{2\eta}{\sigma^2}V;\theta,b\right)$

ALTERNATIVE STOCHASTIC PROCESSES: Investment Characteristics

- Use numerical example with same parameters as before plus $\mu = 0.08$ and varying η and $\overline{V}$
  - As $\overline{V}$ increases, so does the value of waiting and, thus, both $F(V)$ and V^* increase (Figure 5.11)
  - Variation with η: if $\overline{V} > I$, then $F(V)$ increases in η (but decreases otherwise) as V is likely to rise above I and remain there (Figures 5.12 and 5.13)
  - Shape of $F(V)$ becomes concave for small V because the absolute rate of mean reversion rises rapidly
  - V^* increases with η as long as $\overline{V}$ is large (Figure 5.14)

ALTERNATIVE STOCHASTIC PROCESSES: Figure 5.11

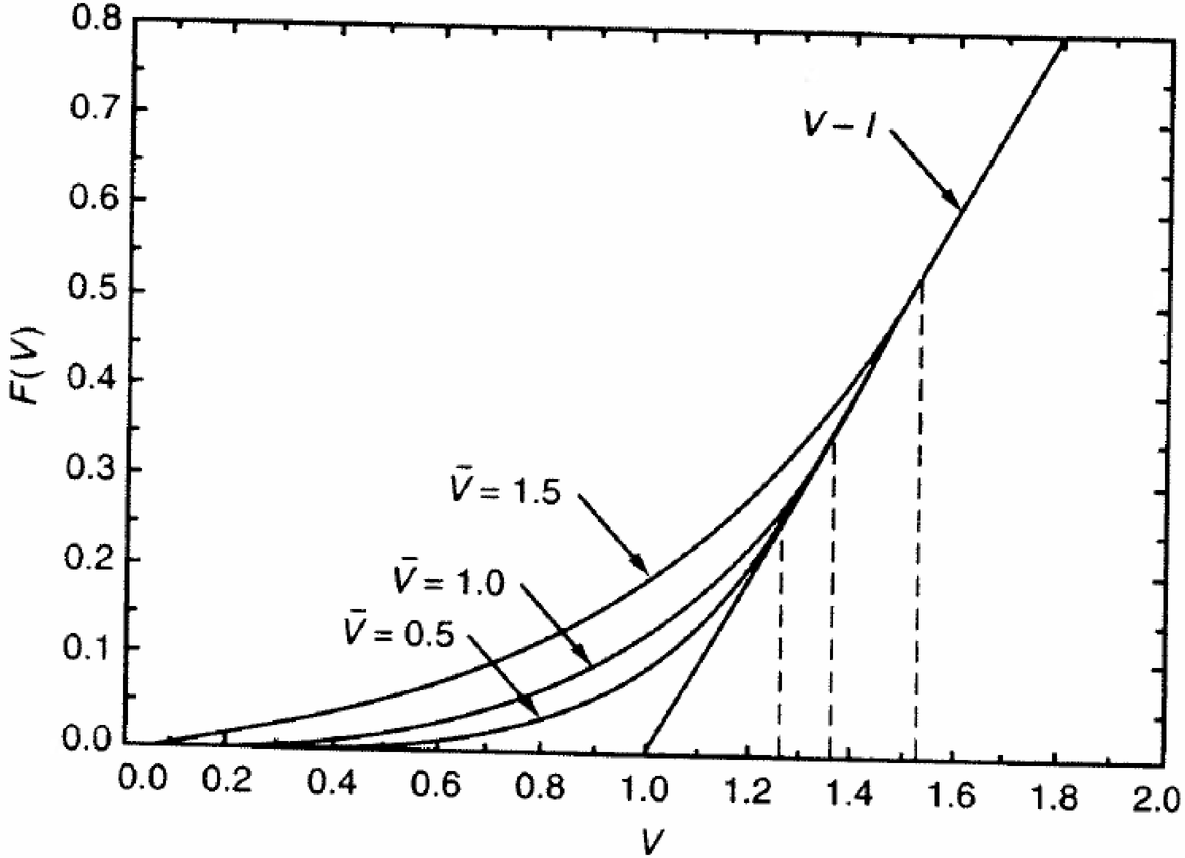


Figure 5.11. *Mean Reversion*—$F(V)$ *for* $\eta = 0.05$ *and* $\bar{V} = 0.5, 1.0,$ *and* 1.5

ALTERNATIVE STOCHASTIC PROCESSES: Figure 5.12

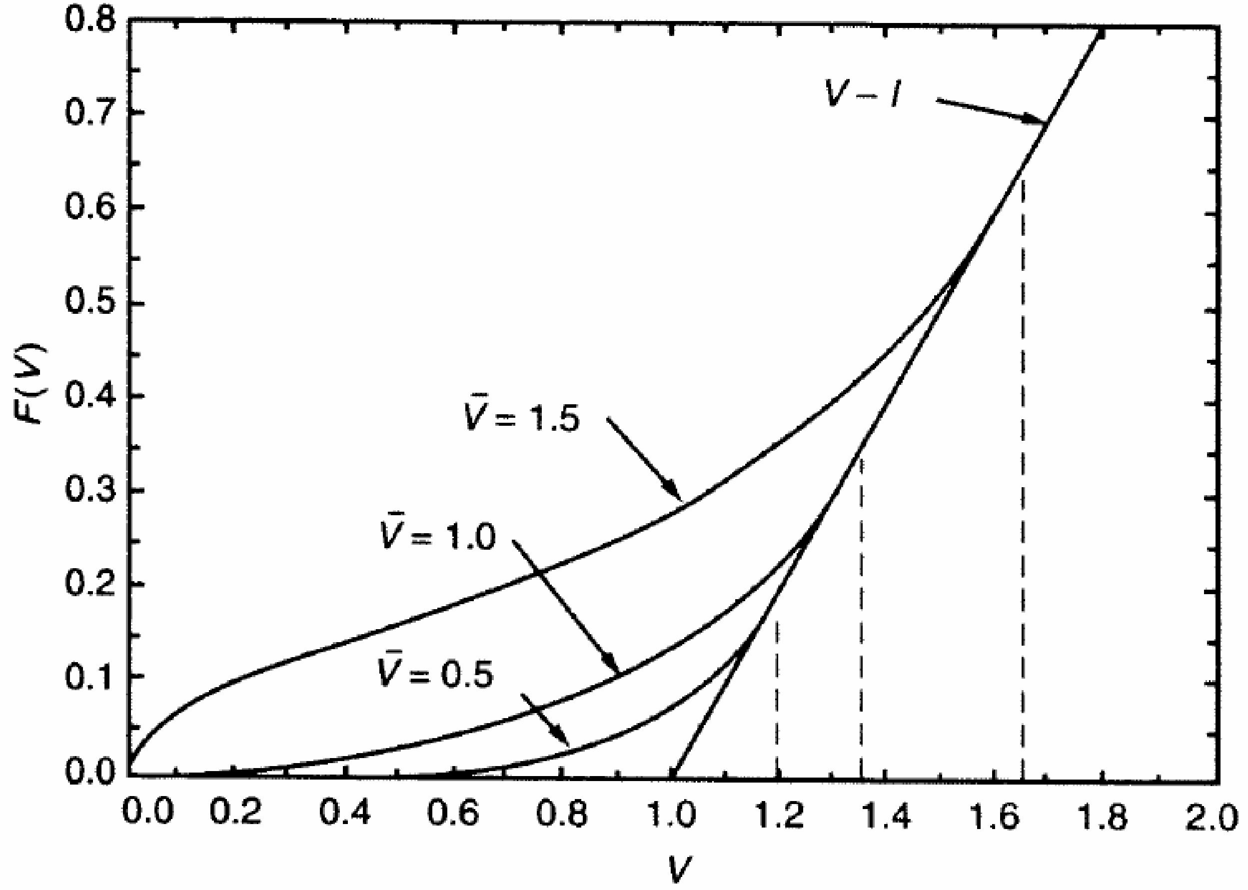


Figure 5.12. *Mean Reversion—$F(V)$ for $\eta = 0.1$ and $\bar{V} = 0.5$, 1.0, and 1.5*

ALTERNATIVE STOCHASTIC PROCESSES: Figure 5.13

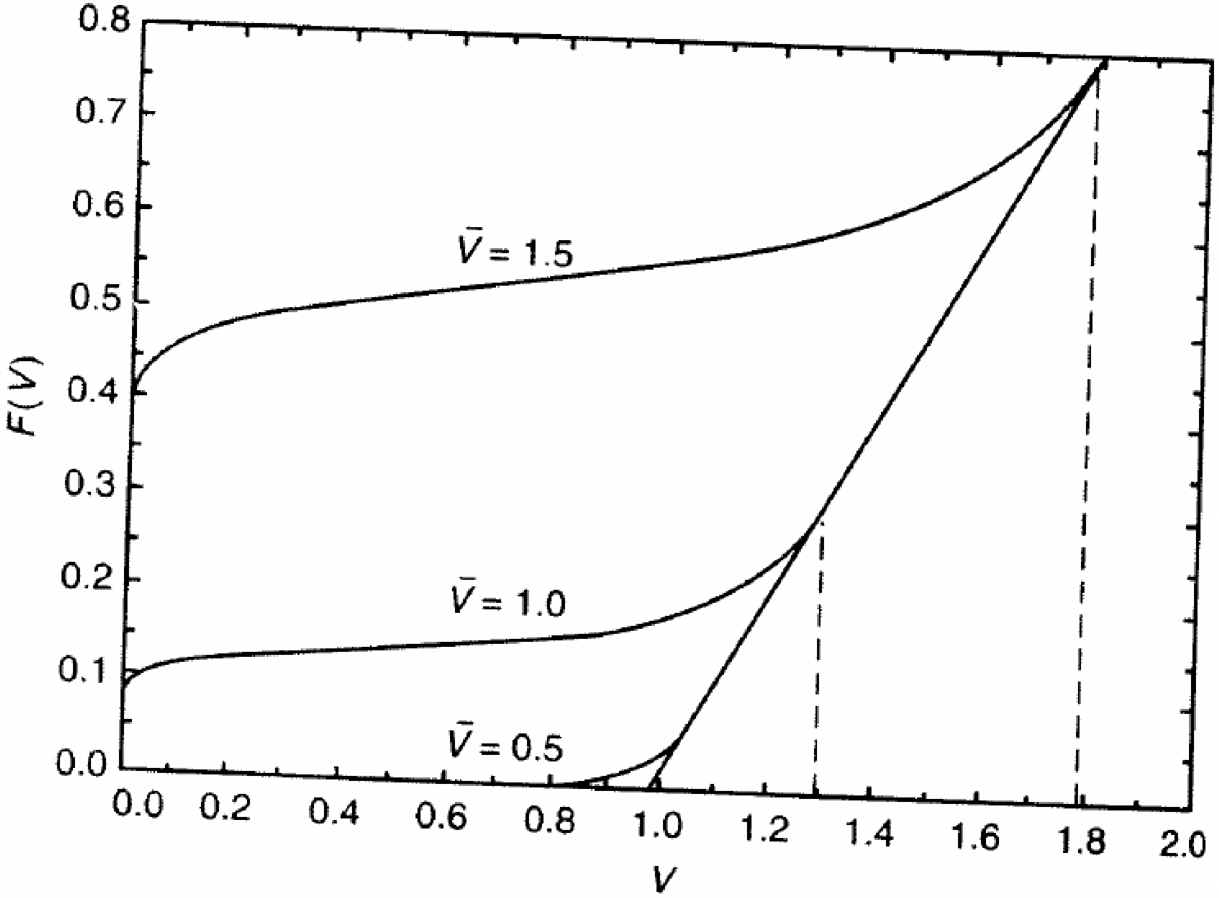


Figure 5.13. *Mean Reversion—$F(V)$ for $\eta = 0.5$ and $\bar{V} = 0.5$, 1.0, and 1.5*

ALTERNATIVE STOCHASTIC PROCESSES: Figure 5.14

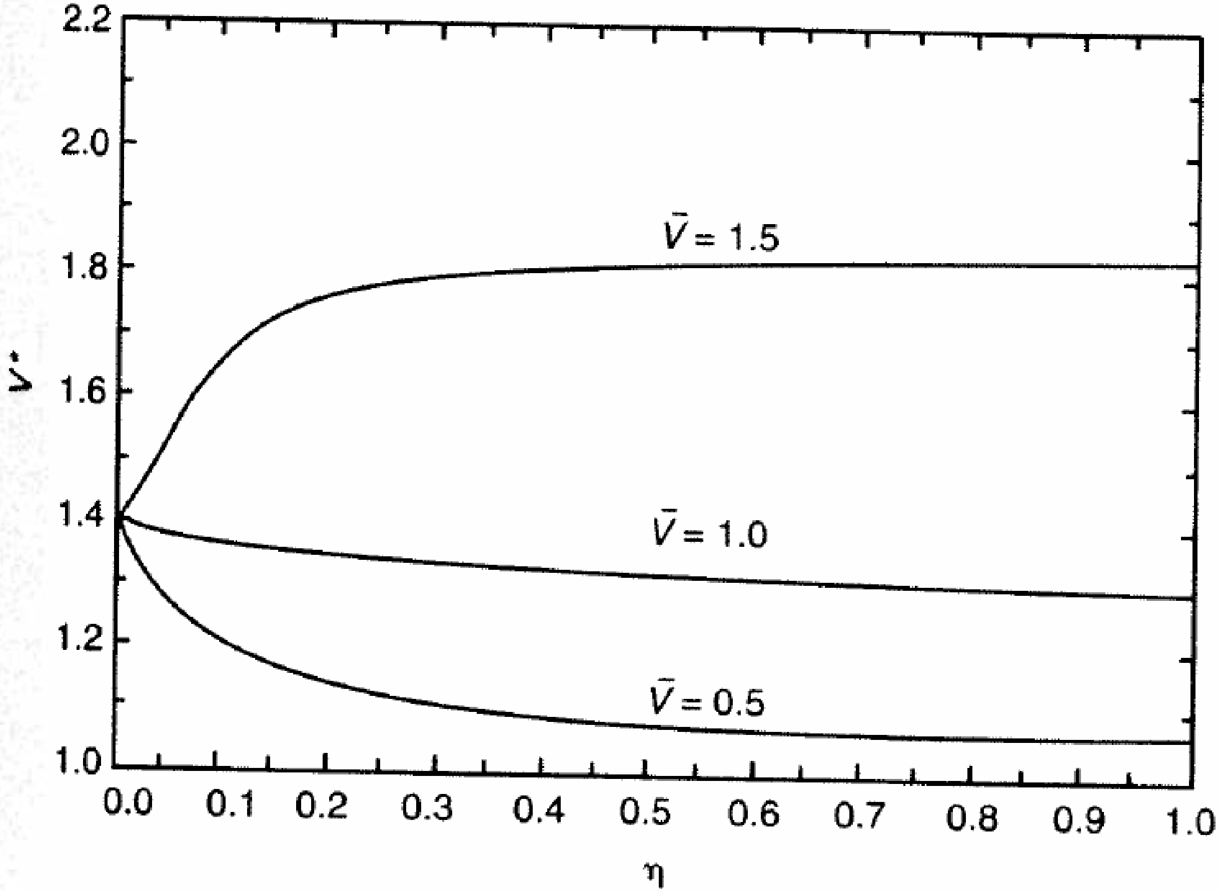


Figure 5.14. *Critical Value V^* as a Function of η for $\mu = 0.08$ and $\bar{V} = 0.5$, 1.0, and 1.5*

VALUE OF THE PROJECT WITHOUT OPERATING COSTS

- Suppose that the output price, P, follows a GBM and the firm produces one unit per year forever
  - Without operating costs and ruling out speculative bubbles, the value of the project is $V(P) = \mathcal{E}_P \int_0^\infty P_t e^{-\mu t} dt = \int_0^\infty \mathcal{E}_P [P_t] e^{-\mu t} dt = \int_0^\infty P e^{-(\mu - \alpha)t} dt = \frac{P}{\delta}$
  - Via the contingent claims argument, we can now find the value of the option to invest, $F(P)$, which will satisfy the ODE $\frac{1}{2}\sigma^2 P^2 F''(P) + (r - \delta) P F'(P) - r F(P) = 0$: $F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}$
  - Boundary condition $F(0) = 0 \Rightarrow A_2 = 0$
  - VM and SP conditions imply: (i)$A_1 (P^*)^{\beta_1} = \frac{P^*}{\delta} - I$ and (ii) $\beta_1 A_1 (P^*)^{\beta_1 - 1} = \frac{1}{\delta}$
  - Therefore, $P^* = \frac{\beta_1}{\beta_1 - 1} \delta I$ and $A_1 = \frac{(\beta_1 - 1)^{\beta_1 - 1} I^{-(\beta_1 - 1)}}{(\delta \beta_1)^{\beta_1}}$
  - Note that $V^* = \frac{P^*}{\delta} = \frac{\beta_1}{\beta_1 - 1} I > I$
- Can also use dynamic programming to find $F(P)$

OPERATING COSTS AND TEMPORARY SUSPENSION: Value of the Project

- Suppose now that the project incurs operating cost, C, but it may be costlessly suspended or resumed once installed
  - Instantaneous profit flow is $\pi(P) = \max[P - C, 0]$, i.e., project owner has infinitely many embedded operational options
  - Thus, the value of an active project will be worth more than simply the NPV of the cash flows
- Value the project using contingent claims by going long one unit $V(P)$ and shorting $n = V_P(P)$ units of P
  - Unlike the option to invest, we now have a profit flow, $\pi(P)$, which implies that the ODE becomes $\frac{1}{2}\sigma^2 P^2 V''(P) + (r - \delta)PV'(P) - rV(P) + \pi(P) = 0$
  - For $P < C$, only the homogeneous part of the solution is valid, i.e., $V(P) = K_1 P^{\beta_1} + K_2 P^{\beta_2}$
  - With $P \geq C$, we also have the particular solution $D_1 P + D_2 C + D_3$
  - Substitution into the ODE yields $D_1 = \frac{1}{\delta}, D_2 = -\frac{1}{r}, D_3 = 0$
  - Therefore, $V(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{r}$ for $P \geq C$

OPERATING COSTS AND TEMPORARY SUSPENSION: Value of the Project

- For $P < C$, $V(P)$ represents the option value of resuming a suspended project
  - Intuitively, this must increase in P and be worthless for very small P
  - Only when $K_2 = 0$ does this hold; thus, $V(P) = K_1 P^{\beta_1}$ for $P < C$
- For $P \geq C$, $V(P)$ is the value of an active project inclusive of the option to suspend operations
  - The suspension option is valuable only for small P and becomes worthless for large P
  - Thus, $B_1 = 0$ and $V(P) = B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{r}$ for $P \geq C$
- Find K_1 and B_2 via VM and SP at $P = C$
  - $K_1 C^{\beta_1} = B_2 C^{\beta_2} + \frac{C}{\delta} - \frac{C}{r}$ and $\beta_1 K_1 C^{\beta_1 - 1} = \beta_2 B_2 C^{\beta_2 - 1} + \frac{1}{\delta}$
  - $K_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{(\beta_2 - 1)}{\delta}\right) > 0$, $B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2}\left(\frac{\beta_1}{r} - \frac{(\beta_1 - 1)}{\delta}\right) > 0$
  - $V(P)$ is increasing (decreasing) in σ (δ) (Figures 6.1 and 6.2)

OPERATING COSTS AND TEMPORARY SUSPENSION: Figure 6.1

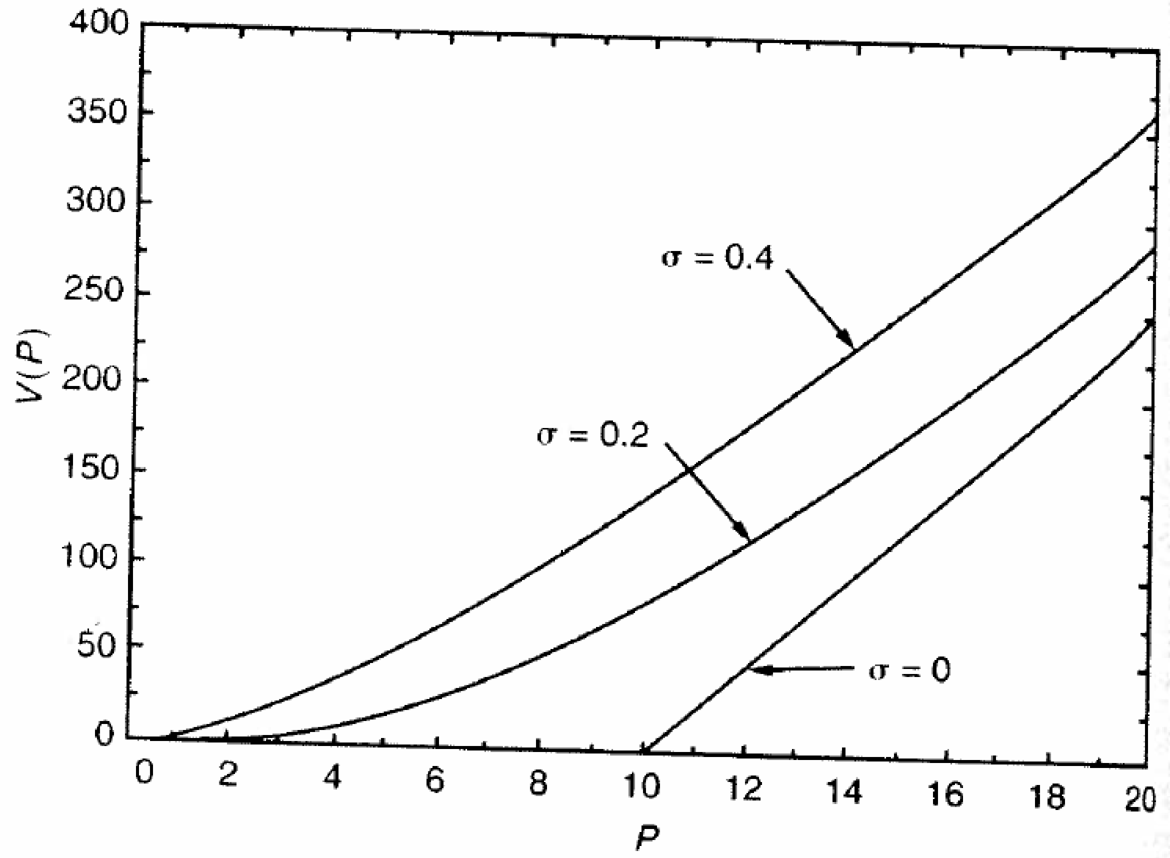


Figure 6.1. *Value of Project, $V(P)$, for $\sigma = 0, 0.2, 0.4$*

(Note: $r = \delta = 0.04$, and $C = 10$)

OPERATING COSTS AND TEMPORARY SUSPENSION: Figure 6.2

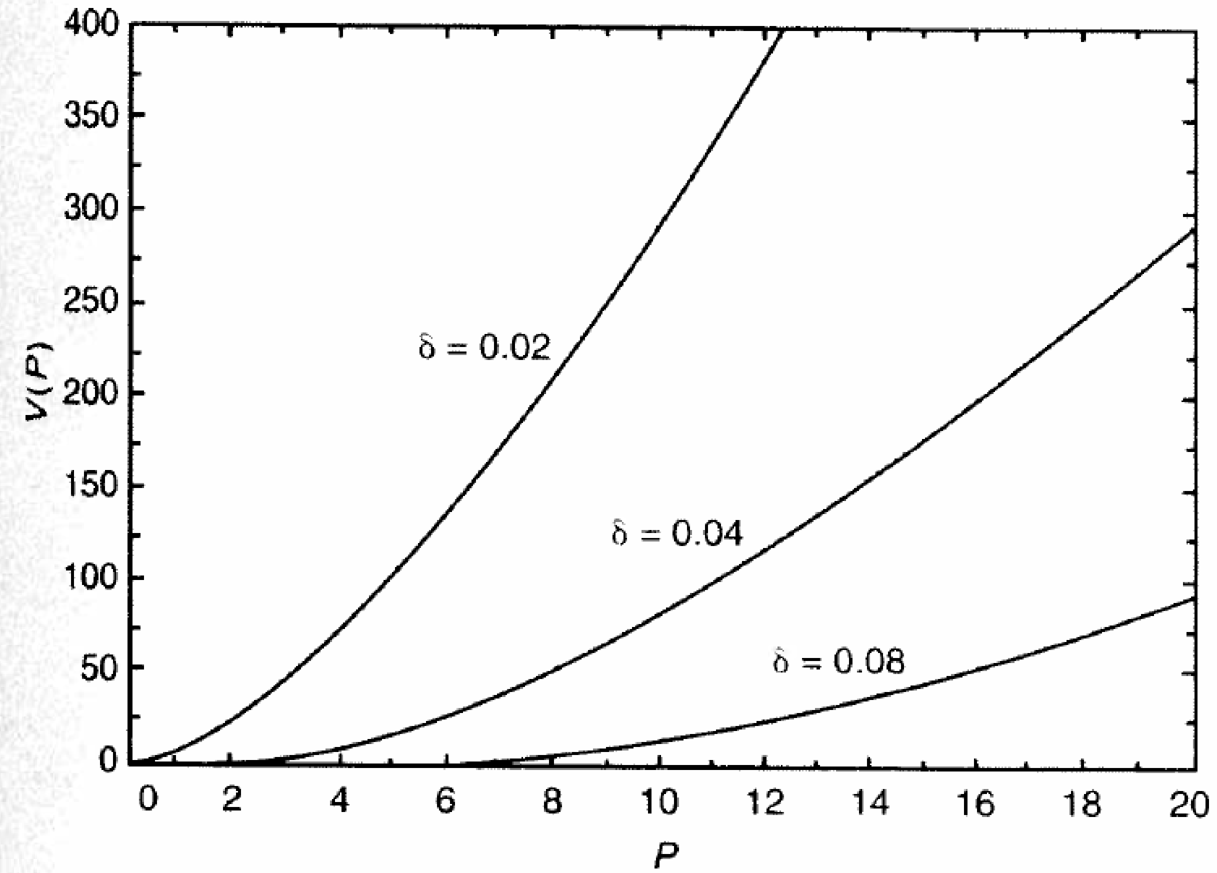


Figure 6.2. *Value of Project, $V(P)$, for $\delta = 0.02, 0.04, 0.08$*

(Note: $r = 0.04$, $\sigma = 0.2$, and $C = 10$)

OPERATING COSTS AND TEMPORARY SUSPENSION: Value of the Option to Invest

- Following the contingent claims approach, $F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2}$
  - Boundary condition $F(0) = 0 \Rightarrow A_2 = 0$
- For $P < C$, it is never optimal to invest
  - Thus, VM and SP of $F(P)$ will occur for $P \geq C$, i.e., with $V(P) - I = B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{r} - I$
  - Use $A_1 (P^*)^{\beta_1} = B_2 (P^*)^{\beta_2} + \frac{P^*}{\delta} - \frac{C}{r} - I$ and $\beta_1 A_1 (P^*)^{\beta_1 - 1} = \beta_2 B_2 (P^*)^{\beta_2 - 1} + \frac{1}{\delta}$ to solve for P^* and A_1
  - Substitute to solve the following equation numerically: $(\beta_1 - \beta_2) B_2 (P^*)^{\beta_2} + (\beta_1 - 1)\frac{P^*}{\delta} - \beta_1 \left(\frac{C}{r} + I\right) = 0$
  - Solution for $r = 0.04$, $\delta = 0.04$, $\sigma = 0.20$, $I = 100$, and $C = 10$ (Figure 6.3)
  - $\beta_1 = 2$, $\beta_2 = -1$, $P^{*,nf} = 28$, $A_1^{nf} = 0.4464$, $P^* = 23.8$, and $A_1 = 0.4943$
  - Sensitivity analysis: $F(P)$ and P^* increase with σ (Figure 6.4)
  - But $F(P)$ decreases and P^* increases with δ (Figures 6.5 and 6.6)

OPERATING COSTS AND TEMPORARY SUSPENSION: Figure 6.3

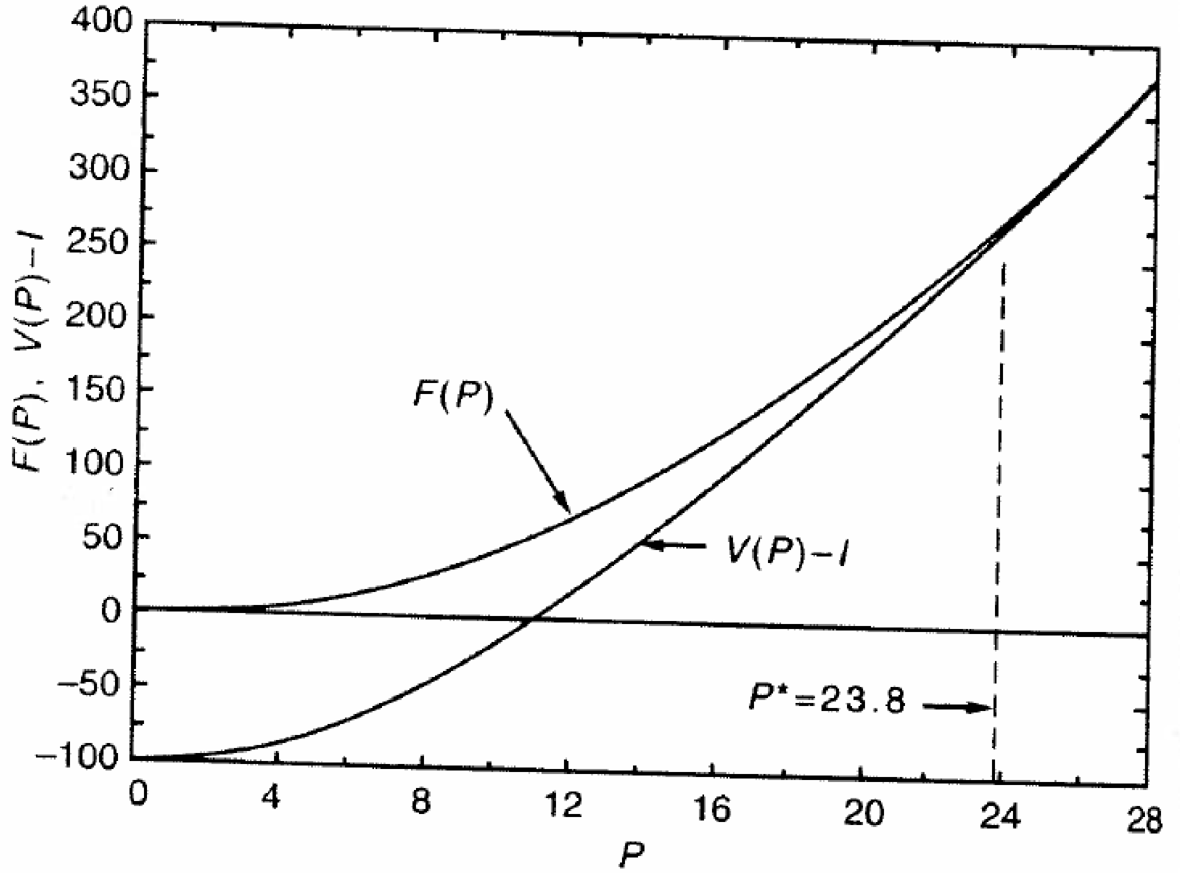


Figure 6.3. Value of Investment Opportunity, $F(P)$, and $V(P)-I$

(Note: $r = \delta = 0.04$, $\sigma = 0.2$, and $I = 100$)

OPERATING COSTS AND TEMPORARY SUSPENSION: Figure 6.4

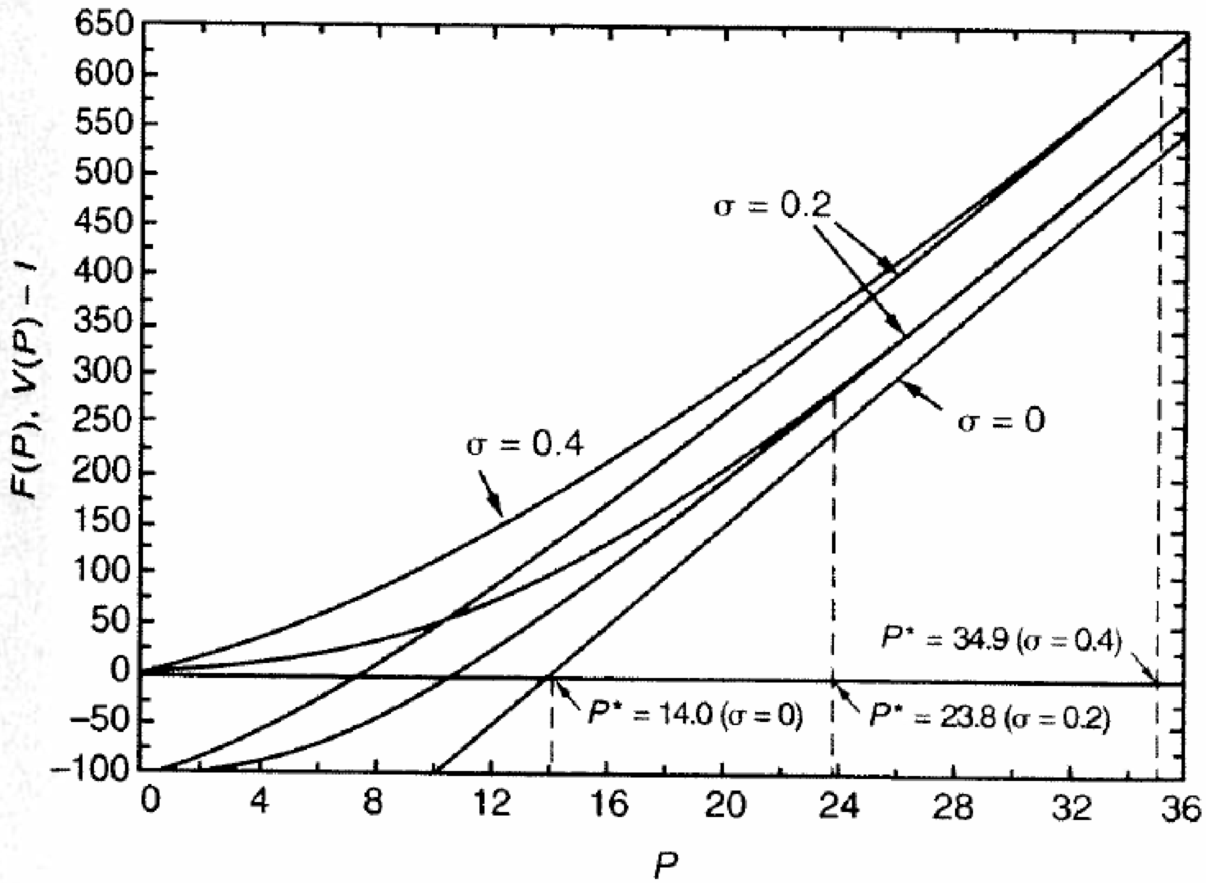


Figure 6.4. *Value of Investment Opportunity, F(P), and V(P) − I, for σ = 0, 0.2, and 0.4*

OPERATING COSTS AND TEMPORARY SUSPENSION: Figure 6.5

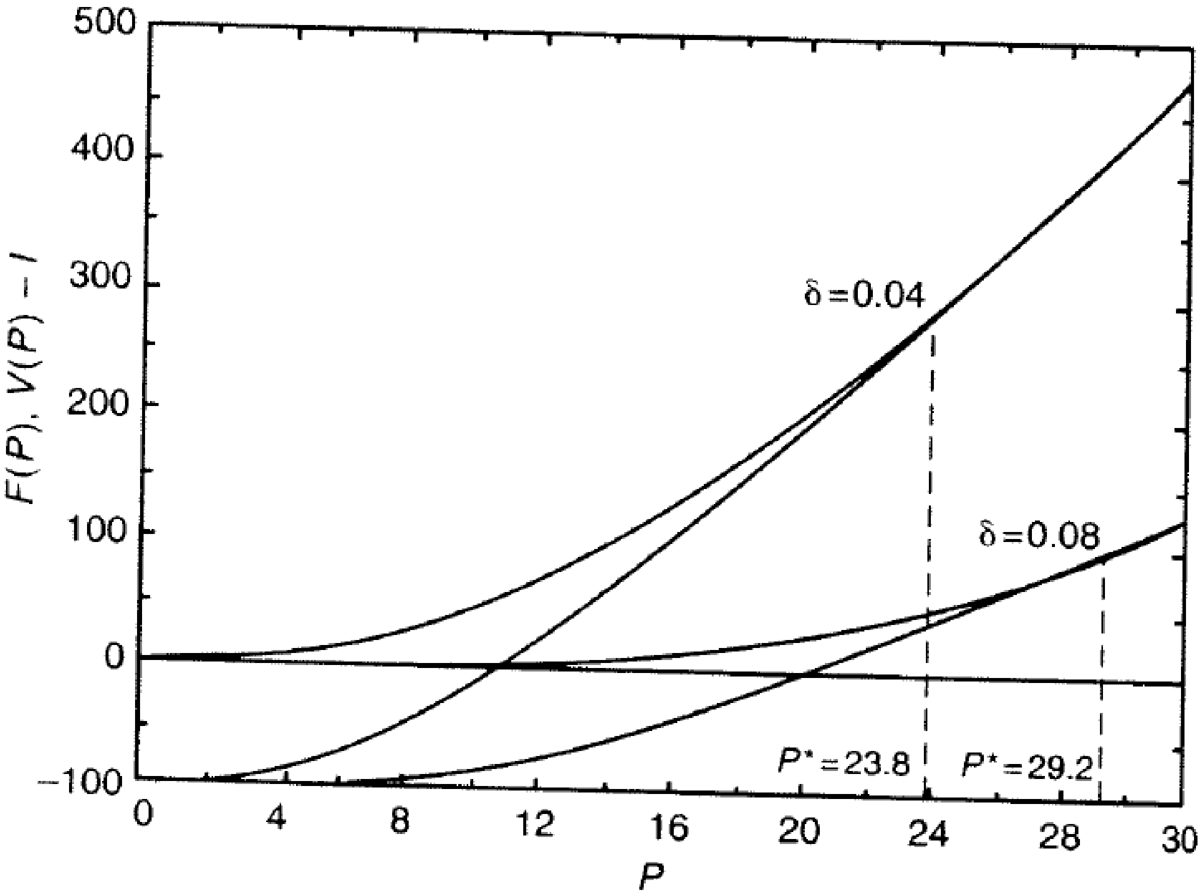
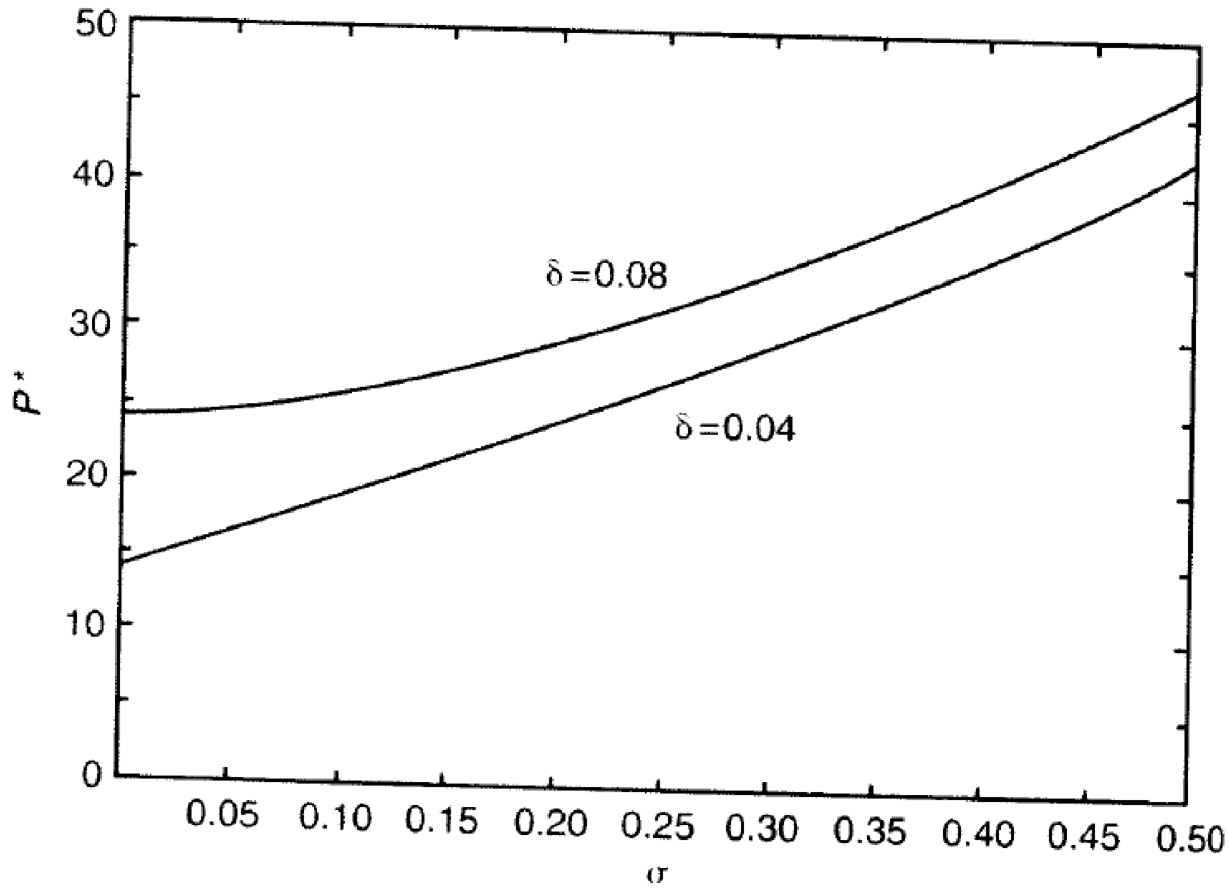


Figure 6.5. *Value of Investment Opportunity, F(P), and V(P) − I, for δ = 0.04 and 0.08*

OPERATING COSTS AND TEMPORARY SUSPENSION: Figure 6.6



PROJECTS WITH VARIABLE OUTPUT: Project Value

- Suppose output is produced according to function $h(v)$, where v is level of some intermediate good
- Instantaneous profit flow is $\pi(P) \equiv \max_v \left[Ph(v) - C(v)\right]$
  - Assume Cobb-Douglas production function, i.e., $h(v) = v^\theta$, where $0 < \theta < 1$, and constant marginal cost, i.e., $C(v) = cv$
  - Profit maximisation yields $v^* = \left[\frac{\theta P}{c}\right]^{\frac{1}{1-\theta}}$ and $\pi(P) = (1 - \theta)\left(\frac{\theta}{c}\right)^{\frac{\theta}{1-\theta}} P^{\frac{1}{1-\theta}}$
  - Let $\gamma \equiv \frac{1}{1-\theta} > 1$ so that $\pi(P) = KP^\gamma$
  - Intuition is that without control, profit changes linearly in price, but variation makes it possible to increase faster (decrease slower) when P rises (falls)
- Standard ODE for the project value is $\frac{1}{2}\sigma^2 P^2 V''(P) + (r - \delta)PV'(P) - rV(P) + KP^\gamma = 0$
  - Guess particular solution of the form $K_1 P^\gamma$ and find that $K_1 = \frac{K}{r-(r-\delta)\gamma-\frac{1}{2}\sigma^2\gamma(\gamma-1)} \Rightarrow V(P) = \frac{KP^\gamma}{\delta'}$

PROJECTS WITH VARIABLE OUTPUT: Option Value

- We require $\delta' > 0$, where δ' is the negative of $\mathcal{Q}(\gamma)$

- Thus, $\delta' > 0 \Leftrightarrow \mathcal{Q}(\gamma) < 0$, which implies that $\gamma < \beta_1$
  - In other words, the production function must have the restriction that $\theta < \frac{\beta_1 - 1}{\beta_1}$

- Solution to the option value is $F(P) = A_1 P^{\beta_1}$
  - Use VM and SP conditions to find $\frac{K(P^*)^\gamma}{\delta'} = \frac{\beta_1}{\beta_1 - \gamma} I$
  - There is a greater incentive to invest now because of the convexity of profit flow: it is possible to benefit from the upside of greater volatility without being hurt by the downside

DEPRECIATION: Exponential Decay with a Single Investment Option

- Suppose that the lifetime of the project, T, follows a Poisson process with parameter λ, i.e., density function is $e^{-\lambda T}$
  - Given that the lifetime is T years, the expected PV of the project is $V_T(P) = \mathcal{E}_P \int_0^T P_t e^{-\mu t} dt = \int_0^T \mathcal{E}_P[P_t] e^{-\mu t} dt = \frac{P}{\delta}\left(1 - e^{-\delta T}\right)$
  - With a random lifetime: $V(P) = \mathcal{E}[V_T(P)] = \int_0^\infty \lambda e^{-\lambda T} \frac{P}{\delta}\left(1 - e^{-\delta T}\right) dT = \frac{P}{\lambda + \delta}$
  - Project functions less well over time, which eats into its cash flows

- Value of option to invest may be obtained using contingent claims: $F(P) = A_1 P^{\beta_1}$

- VM and SP conditions reveal $P^* = \frac{\beta_1}{\beta_1 - 1}(\delta + \lambda) I$

DEPRECIATION: Exponential Decay with Re-investment

- Upon termination, re-investment is available at cost I
- If no investment has occurred, then the option value to invest is again $F(P) = A_1 P^{\beta_1}$
- Let $J(P)$ be the value of an active project along with all future replacement options (use dynamic programming)
  - When $P < P^*$, there is a profit flow and probability λdt that the project will die in the next dt time units
  - Conditional expectation: $J(P) = Pdt + (1 - \lambda dt)e^{-\rho dt}\mathcal{E}[J(P + dP)] + \lambda dt e^{-\rho dt}\mathcal{E}[F(P + dP)]$
  - Note that $\mathcal{E}[J(P+dP)] = J(P) + J'(P)\alpha P dt + \frac{1}{2}J''(P)\sigma^2 P^2 dt$ and $\mathcal{E}[F(P + dP)] = F(P) + F'(P)\alpha P dt + \frac{1}{2}\sigma^2 P^2 F''(P)dt$
  - Thus, $J(P) = Pdt + (1 - (\rho + \lambda)dt)[J(P) + J'(P)\alpha P dt + \frac{1}{2}\sigma^2 P^2 J''(P)dt] + \lambda dt A_1 P^{\beta_1}[1 + \alpha\beta_1 dt + \frac{1}{2}\sigma^2\beta_1(\beta_1 - 1)dt] \Rightarrow \frac{1}{2}\sigma^2 P^2 J''(P) + \alpha P J'(P) - (\rho + \lambda)J(P) + \lambda A_1 P^{\beta_1} + P = 0$
  - Solution is $J(P) = B_1 P^{\beta_1'} + \frac{P}{\rho+\lambda-\alpha} + A_1 P^{\beta_1}$, where β_1' is the positive root of $\frac{1}{2}\sigma^2\xi(\xi - 1) + \alpha\xi - (\rho + \lambda) = 0$

DEPRECIATION: Exponential Decay with Re-investment

- For $P \geq P^*$, re-investment is immediate upon termination
  - Conditional expectation: $J(P) = Pdt + (1 - \lambda dt)e^{-\rho dt}\mathcal{E}[J(P + dP)] + \lambda dt e^{-\rho dt}\mathcal{E}[J(P + dP) - I]$
  - Thus, $J(P) = Pdt + (1 - (\rho + \lambda)dt)[J(P) + J'(P)\alpha P dt + \frac{1}{2}\sigma^2 P^2 J''(P)dt] + \lambda dt J(P) - \lambda I dt \Rightarrow \frac{1}{2}\sigma^2 P^2 J''(P) + \alpha P J'(P) - \rho J(P) + P - \lambda I = 0$
  - Solution is $J(P) = B_2 P^{\beta_2} + \frac{P}{\rho - \alpha} - \frac{\lambda I}{\rho}$
  - Two branches of $J(P)$ meet tangentially at P^* and have the usual VM and SP conditions with $F(P)$

- Find $P^* = \frac{\beta_1'}{\beta_1' - 1}(\delta + \lambda)I$, i.e., lower investment threshold than when only a single option was available

PRICE AND COST UNCERTAINTY

★ Both P and I follow correlated GBMs

- $dP = \alpha_P P dt + \sigma_P P dz_P$, $dI = \alpha_I I dt + \sigma_I I dz_I$, $\mathcal{E}[(dz_P)^2] = dt$, $\mathcal{E}[(dz_I)^2] = dt$, and $\mathcal{E}[dz_P dz_I] = \rho dt$
- Expected NPV of project is $V(P, I) = \frac{P}{\delta_P} - I$, and we want $F(P, I)$
- Construct risk-free portfolio: $\Phi = F - n_P P - n_I I \Rightarrow d\Phi = dF - n_P dP - n_I dI$
- $dF = F_P dP + F_I dI + \frac{1}{2} F_{PP}(dP)^2 + \frac{1}{2} F_{II}(dI)^2 + F_{PI}(dPdI) \Rightarrow dF = F_P dP + F_I dI + \frac{1}{2} F_{PP}\sigma_P^2 P^2 dt + \frac{1}{2} F_{II}\sigma_I^2 I^2 dt + F_{PI}\sigma_P\sigma_I PI\rho dt$
- Substitution implies $d\Phi = (F_P - n_P)dP + (F_I - n_I)dI + \frac{1}{2} F_{PP}\sigma_P^2 P^2 dt + \frac{1}{2} F_{II}\sigma_I^2 I^2 dt + F_{PI}\sigma_P\sigma_I PI\rho dt$
- In order for Φ to be risk free, we must have $n_P = F_P$ and $n_I = F_I$
- Add the convenience yield to obtain the total portfolio return: $\frac{1}{2} F_{PP}\sigma_P^2 P^2 dt + \frac{1}{2} F_{II}\sigma_I^2 I^2 dt + F_{PI}\sigma_P\sigma_I PI\rho dt - F_P\delta_P P dt - F_I\delta_I I dt$
- Risk-free rate of return: $r\Phi dt = rFdt - rF_P P dt - rF_I I dt$
- Obtain PDE: $\frac{1}{2} F_{PP}\sigma_P^2 P^2 + \frac{1}{2} F_{II}\sigma_I^2 I^2 + F_{PI}\sigma_P\sigma_I PI\rho + (r - \delta_P)F_P P + (r - \delta_I)F_I I - rF = 0$
- VM: $F(P^*(I), I) = \frac{P^*(I)}{\delta_P} - I$, SP1: $F_P(P^*(I), I) = \frac{1}{\delta_P}$, and SP2: $F_I(P^*(I), I) = -1$

PRICE AND COST UNCERTAINTY

★ Use transformation to convert PDE to ODE

- Let $p = \frac{P}{I}$ and $f(p) = \frac{F(P,I)}{I}$
- Thus, $F_P = f'(p)$, $F_I = f(p) - pf'(p)$, $F_{PP} = f''(p)I^{-1}$, $F_{II} = \frac{p^2 f''(p)}{I}$, and $F_{PI} = -\frac{pf''(p)}{I}$
- The ODE is $\frac{1}{2}\left(\sigma_P^2 - 2\rho\sigma_P\sigma_I + \sigma_I^2\right)p^2 f''(p) + (\delta_I - \delta_P)pf'(p) - \delta_I f(p) = 0$
- VM: $f(p^*) = \frac{p^*}{\delta_P} - 1$ and SP: $f'(p^*) = \frac{1}{\delta_P}$
- Therefore, $f(p) = a_1 p^{\gamma_1}$, where γ_1 is the positive root of $\frac{1}{2}\left(\sigma_P^2 - 2\rho\sigma_P\sigma_I + \sigma_I^2\right)\beta(\beta - 1) + (\delta_I - \delta_P)\beta - \delta_I = 0$
- Thus, $p^* = \frac{\gamma_1}{\gamma_1 - 1}\delta_P$
- In other words, higher uncertainty causes the free boundary to rotate upwards (Figure 6.8)
- What happens when ρ is increased?

PRICE AND COST UNCERTAINTY: Figure 6.8

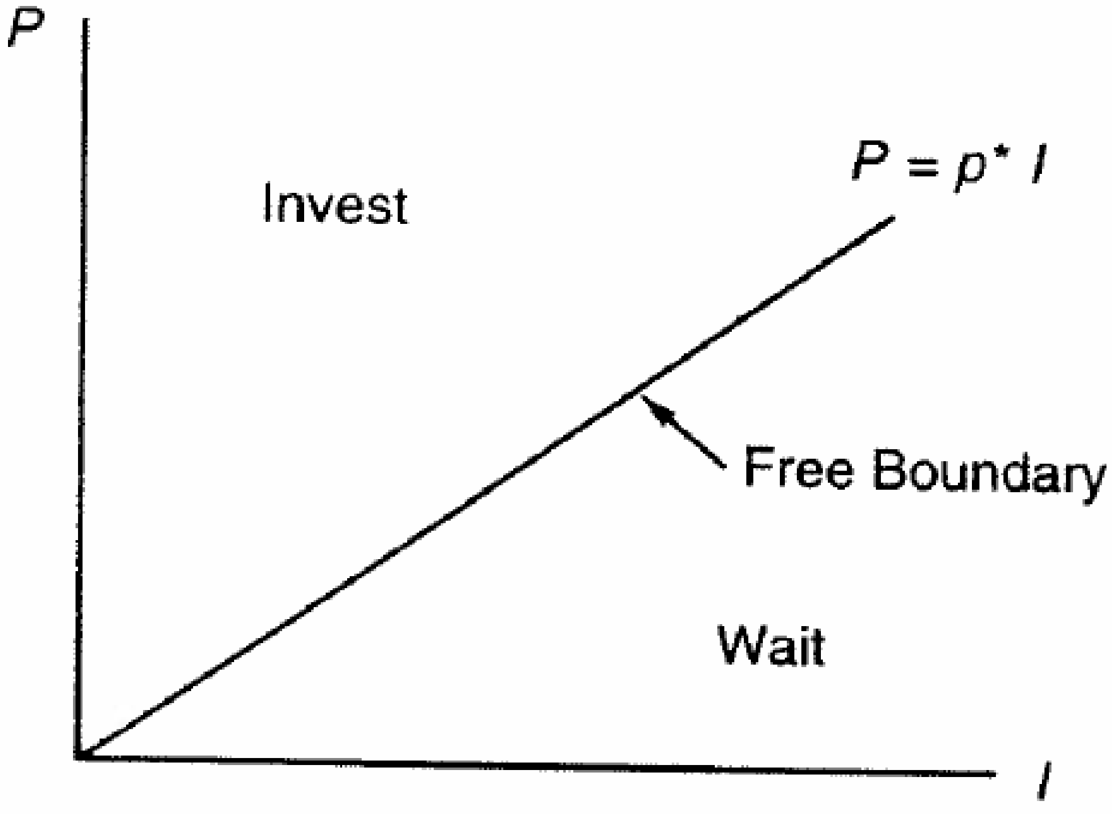


Figure 6.8. *Investment with Price and Cost Uncertainty*

QUESTIONS