AALTO UNIVERSITY
Systems analysis laboratory

Mat-2.4136 Special Topics in Decision Making: Aggregation functions
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Exercise 1: General properties

1. Assume the convention $0/0 = 0$. Is $\mathsf{A}(x_1, x_2) = \frac{x_1^2+x_2^2}{x_1+x_2}$ an aggregation function in $[0, 1]$? Justify your answer.

2. Consider that

   - $x \in \mathbb{R}$ is an *idempotent element* of F if and only if $\mathsf{F}(x, \ldots, x) = x$.
   - F is *unanimously increasing* if and only if

   $$x_i > y_i \ \forall i \Rightarrow \mathsf{F}(x_1, \ldots, x_n) > \mathsf{F}(y_1, \ldots, y_n).$$

   Then fill the following table

	idemp elements	associative	bisymmetric	unanim. increas.
$\mathsf{max}(\mathbf{x}) = \max\{x_1, \ldots, x_n\}$	$x \in [0, 1]$	YES	YES	YES
$\mathsf{median}(\mathbf{x}) = median(x_1, \ldots, x_n)$				
$\mathsf{HM}(\mathbf{x}) = \frac{n}{\sum_{i=1}^{n} 1/x_i}$				
$\min\{\sum_{i=1}^{n} x_i, 1\}$				
$\mathsf{M}_{0.5}(\mathbf{x}) = median(0, 0.5 + \sum_{i=1}^{n}(x_i - 0.5), 1)$				

3. A *lattice polynomial* is an expression involving a number of variables $x_1, x_2, \ldots,$ logical operators $\wedge$, $\vee$ and parentheses. For instance

   $$(x_1 \vee x_2) \wedge x_3 \wedge x_1$$

   is a lattice polynomial. Write down the median function of three values, x_1, x_2, x_3 as a lattice polynomial.
   Note: It is possible to interpret $\vee = \max$ and $\wedge = \min$ so that the lattice polynomial above would be $\min\{\max\{x_1, x_2\}, x_3, x_1\}$

4. The product $\Pi(\mathbf{x}) = \prod_{i=1}^{n} x_i$ is an aggregation function in $[0, 1]$. Consider the extended real line $\bar{\mathbb{R}} = [-\infty, +\infty]$ with its operations, e.g. $+\infty \cdot 0 = 0$ and $+\infty \cdot +\infty = +\infty$. Then the product is also an aggregation function on $[0, +\infty]$. On what other subset of $[-\infty, +\infty]$ is the product defined as an aggregation function?

5. Consider an associative and idempotent extended aggregation function. Is it variant under repetition of components of $\mathbf{x}$? Prove it.

6. Consider the Einstein sum

   $$\mathsf{E}(x_1, x_2) = \frac{x_1 + x_2}{1 + x_1 x_2}$$

   in example 1.73. Use the associativity property to extend it to the case of three arguments x_1, x_2, x_3.