AALTO UNIVERSITY
Systems analysis laboratory

Mat-2.4136 Special Topics in Decision Making: Aggregation functions
Matteo Brunelli

Exercise 2: Means

1. Prove that the arithmetic mean (AM), the harmonic mean (HM) and the geometric mean (GM) satisfy the following relation:
$$(\mathsf{GM}(\mathbf{x}))^n = \mathsf{HM}(\mathbf{x}) \cdot \mathsf{AM}(x_1 x_2 \cdots x_{n-1}, \ldots, x_2 x_3 \cdots x_n)$$

2. You have a vector of observations $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Find the analytic solution to this fitting problem
$$\arg\min_x \sum_{i=1}^{n} (\ln x - \ln x_i)^2$$
Basically you want to find which real number minimizes the sum of the squares of the logarithmic differences. Do the same for
$$\arg\min_x \sum_{i=1}^{n} \left(\frac{1}{x} - \frac{1}{x_i}\right)^2$$
Hint: They can be solved by using their derivatives in x (basic unconstrained optimization).

3. The k*th ordered statistic* is simply the kth greatest element of a vector $\mathbf{x}$ and it is written $\mathsf{OS}_k(\mathbf{x})$ (def. 2.161). It is an aggregation function. Interestingly it can also be written as a lattice polynomial. Find the right suffixes for the operators $\wedge$ and $\vee$ in the general case so that it would work for any $k \leq n$ and $n \in \{2, 3, \ldots\}$.
$$\mathsf{OS}_k(\mathbf{x}) = \bigvee_{?} \bigwedge_{?} x_i$$
Hint: recall the exercise on the median of three values x_1, x_2, x_3 and reckon that this is a more general case.

4. Use the notation
$$\mathsf{M}_f := f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$$
and $\mathbb{I}$ that is an interval where M_f and M_g are defined on. Then prove the following. Let $f, g : \mathbb{I} \to \mathbb{R}$ be continuous and strictly monotone functions: Assume also that g is increasing (respectively, decreasing). Then,

   (a) $\mathsf{M}_f \leq \mathsf{M}_g$ if and only if $g \circ f^{-1}$ is convex (respectively, concave);
   Hint: Start with $\mathsf{M}_f \leq \mathsf{M}_g$ and report it to a form of the Jensen inequality (see MathWorld)

(b) $\mathsf{M}_f = \mathsf{M}_g$ if and only if $g \circ f^{-1}$ is linear, i.e. $g(x) = rf(x) + s$, for all $r, s \in \mathbb{R}, r \neq 0$.

5. There is an offshore oilfield where a platform A is already installed and it is estimated that it would take other 4 years to extract all the oil. The company is thinking to add a smaller and slower (but very cheap) new platform B. If B worked alone, then the extraction would be completed in 10 years. Fortunately, the two facilities can work in parallel. How long would it take them, in this case, to finish the extraction if they work together? Justify your result.
   Hint: the result is something similar to the harmonic mean.

6. Consider the family of power means. Show that, with $x_i > 0 \; \forall i$

$$\lim_{r \to 0} \left(\frac{1}{n} \sum_{i=1}^{n} x_i^r \right)^{\frac{1}{r}} = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}$$

   Hint: google "The L'Hôpital Rule: Deriving the Geometric Mean"