AALTO UNIVERSITY
Systems analysis laboratory

Mat-2.4136 Special Topics in Decision Making: Aggregation functions
Matteo Brunelli

Exercise 3: Ordered weighted averaging (OWA) functions

1. Consider the following three candidates evaluated by four judges

	J1	J2	J3	J4
C_1	0.9	0.8	0.3	0.5
C_2	0.2	0.7	0.7	0.7
C_3	0.6	0.6	0.6	0.7

   Evaluate them by means of OWA functions with weight vectors $\mathbf{w}_1 = (0, 1/2, 1/2, 0)$ and $\mathbf{w}_2 = (1/8, 1/8, 2/8, 1/2)$.

   Calculate the orness and the dispersion (entropy) for both $\mathbf{w}_1$ and $\mathbf{w}_2$.

   Remember

$$orness(\mathbf{w}) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i \qquad\qquad disp(\mathbf{w}) = -\sum_{i=1}^{n} w_i \log w_i.$$

2. Use the quantifier $Q(x) = x^3$ to construct the 5-ary weight vector for an OWA function.

3. Let $Q : [0,1] \to [0,1]$ be a monotone increasing bijection with $Q(0) = 0$ and $Q(1) = 1$. We call Q a 'quantifier'. Consider the vector $\mathbf{w} = (w_1, \ldots, w_n)$ whose components are obtained as

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)$$

   Show that $\mathbf{w}$ obtained in such a way is a weight vector, i.e. $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. Hint: the proof is very easy.

4. Consider the two quantifiers Q and Q', and let $\mathbf{w}$ and $\mathbf{w}'$ be the weight vectors obtained by means of Q and Q' respectively. Prove that

$$Q(y) \geq Q'(y)\ \forall y \in [0,1] \Rightarrow orness(\mathbf{w}) \geq orness(\mathbf{w}')$$

5. Take again the two quantifiers Q and Q', and let $\mathbf{w}$ and $\mathbf{w}'$ be the weight vectors obtained by means of Q and Q' respectively. Consider the following two statements:

   (a) $\int_{y\in[0,1]} Q(y)\,dy \geq \int_{y\in[0,1]} Q'(y)\,dy \Rightarrow orness(\mathbf{w}) \geq orness(\mathbf{w}')$

   (b) $w_1 \geq w_2 \geq \cdots \geq w_n \Rightarrow orness(\mathbf{w}) \geq 0.5$

   Claim and prove their truth, or falsity. Hint: one is true, and I suspect the other one to be false.

6. Consider the formula of entropy of a weight vector here called $disp(\mathbf{w})$. Prove that the vector maximizing it is $\mathbf{w} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. It is the translation in our framework of the well-known fact, in information theory, that the uniform distribution is the one maximizing entropy.
   Hint: this is the most famous application of the Gibbs' inequality. Here, Gibbs' inequality would state that, given two weight vectors $(w_1, \ldots, w_n)$ and $(v_1, \ldots, v_n)$, the following is true
   $$\sum_{i=1}^{n} w_i \log\left(\frac{v_i}{w_i}\right) \leq 0$$
   Also, try to start the proof with $disp(\mathbf{w}) - \log(n)$ and see if you can reach the Gibbs' inequality.